ON THE T-NATURE OF DIFFERENTIAL INEQUALITIES

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It is proved that the linear hull of a nonsmooth, in general, T-system is, under a special regularity condition, a generalized ET-space.

In the classical theory of scalar differential equations the following results, which are due to Poincaré [1], play a special role. If a differential operator Lx of order n does not oscillate on the closed interval [a, b], any solution of the differential inequality $Lx > 0$ will have on [a, b] at most n zeros (including multiplicities). This property, which has been applied for the case of residues in the theory of interpolation polynomials (when $Lx \equiv x^{(n)}$) plays a basic role in the theory of de la Vallée—Poussin type multi-point boundary-value problems. Sign-theoretic and sign-regular properties of Green's function and estimates of Green's function, the Sturm properties of the spectrum, and a number of other important results have been established on the basis of this property (cf. [2-4]).

New classes of boundary-value problems that arise in the study of oscillations of elastic systems in which the solution obviously and strongly lacks smoothness, since it is pasted together out of a finite number of smooth solutions (as if from spline functions) have now been identified. It therefore is necessary to prove analogous properties without making any assumption regarding the smoothness of the particular functions. The main property we are interested in will sound something like this: If x(t) is a T-continuation of the T-system $\{\varphi_i\}_0^n$ the total multiplicity of the zeros of x(t) does not exceed n + 1 [the solution x(t) of the differential inequality $Lx > 0$ is a T-continuation of the fundamental system of solutions of the differential equation $Lx = 0$]. A discussion of this question was begun in [5].

In the present article we will develop the methods and results of [5], emphasizing the properties of a T-continuation. These properties are essential in the study of the oscillatory properties of solutions of, for example, quasidifferential inequalities or, what is, in some sense, equivalent, the positive invertibility of nonstandard boundary-value problems with piecewise-smooth solutions.

1. Suppose that H is a finite-dimensional linear space of functions that are continuous on the interval (a, b). Let us say that H is not right-oscillatory at the point $\xi \in [a, b)$, if for some $\varepsilon > 0$, none of the functions $x \in H$, $x(t) \not\equiv 0$ has zeros on the interval $(\xi, \xi + \varepsilon)$. Left nonoscillatoriness at the point ξ for $\xi \in (a, b]$ is defined analogously. We will say that H is nonoscillatory on some subset $J \subset [a, b]$ if H is neither left- nor right-oscillatory at every point $\xi \in J$.

Examples of such H include the set of polynomials in the Chebyshev system or the solution set of a linear homogeneous differential equation without singularities in the coefficients.

Suppose that n is the dimension of H. Following [5], we will call the system $\Phi_n = \{\varphi_i\}_0^{n-1} \subset H$ a right basis of the point $\xi \in [a, b)$, if $\varphi_i(t) = o(\varphi_{i-1}(t))$, $i = \overline{1, n-1}$ as $t \downarrow \xi$ (i.e., $t \to \xi + 0$). A left basis of any point $\xi \in (a, b]$ may be defined in an analogous way. Obviously, any basis is linearly independent, i.e., is a basis in H.

<u>THEOREM 1.</u> Suppose that H is not right- (left-) oscillatory at the point ξ. Then there exists a right (left) basis of ξ.

<u>Proof.</u> Suppose that H is not right-oscillatory at the point ξ. Then for any $x, y \in H$, $y(t) \not\equiv 0$, there exists a finite or infinite right limit of the ratio $x(t)/y(t)$ at ξ (mutual differentiability property). In fact, suppose that $z(t) = x(t)/y(t) \not\equiv \text{const}$. Since y(t) is

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not right-oscillatory at ξ, for some $\varepsilon > 0$, the function $z(t)$ is continuous on $(\xi, \xi + \varepsilon)$. Let us suppose the contrary, i.e., that $z(t)$ does not have any limit as $t \downarrow \xi$. But then there exists a number α and sequence $t_n \downarrow \xi$ such that $z(t_n) = \alpha$, i.e., all points $t_n \in (\xi, b)$ are zeros of the function $x(t) - \alpha\, y(t) \in H$, which contradicts the fact that the function is not right-oscillatory at ξ. The contradiction proves that there exists for $z(t)$ a finite or infinite limit. Let us now show that there exists for the point ξ a right basis. Suppose that $F_n = \{f_i(t)\}_0^{n-1}$ is some basis in H, i.e., $E(F_n) = H$. The function $\varphi_0(t)$ is selected so that $|\lim_{t\to\xi} x(t)/\varphi_0(t)| < \infty$ for any $x(t) \in H$.

Such a function exists since by the mutual differentiability property a number $A_{ij} = \lim_{t\downarrow\xi} f_i(t)/f_j(t)$, $i, j = \overline{0, n-1}$, finite or infinite, may always be defined. A partial ordering may be introduced on the index set $\mathfrak{N} = \{0, 1, \ldots, n - 1\}$ by setting $i < j$ if A_{ij} is finite. Obviously, this relation is defined on any pair $i, j \in \mathfrak{N}$ since if A_{ij} is infinite, then $A_{ij} = 0$, i.e., $i > j$. Moreover, $A_{ij} = A_{ik}A_{kj}$ for any $i, j, k \in \mathfrak{N}$. But then a maximal element k_0 may be found such that the numbers A_{ik_0} are finite for all $i \in \mathfrak{N}$.

Now let us prove that $\varphi_0(t) = f_{k_0}(t)$. We will then consider the space $H_{n-1} \subset H$;

$$H_{n-1} = E(F^1), \quad F^1 = \{f_i^1(t)\}, \quad i = \mathfrak{N} \setminus k_0, \quad f_i^1(t) = f_i(t) - A_{ik_0}\varphi_0(t).$$

Obviously, H_{n-1} consists of functions $x \in H$ such that $x(t) = o(\varphi_0(t))$ as $t \downarrow \xi$ and H_{n-1} is not right-oscillatory at ξ. This means that all the preceding arguments may be repeated and a function $\varphi_1(t)$ thereby identified such that $|\lim_{t\downarrow\xi} x(t)/\varphi_1(t)| < \infty$ for any $x(t) \in H_{n-1}$. After n steps we obtain a sequence $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ such that $\varphi_k(t) = o(\varphi_{k-1}(t))$, $t \downarrow \xi$, $k = \overline{1, n-1}$. That is, $\Phi_n = \{\varphi_i(t)\}_0^{n-1}$ is the desired basis. It may be proved by analogous arguments that there exists a left basis for a right-nonoscillatory H at the point ξ.

<u>Remark.</u> The converse assertion is, in general, false, i.e., that there exists a right (left) basis at the point ξ is no guarantee that H is right (left) nonoscillatory at the point ξ.

Here is a relevant example. Consider the linear hull of the system $\Phi_3 = \{1, t\,\mathrm{Sin}(1/t), t^2\,\mathrm{Sin}(1/t)\}$ on the closed interval $[-1, 1]$. Obviously, Φ_3 is a zero basis, but φ_1 and φ_2 have infinitely many zeros in any neighborhood of Φ_3, i.e., H is oscillatory at 0.

2. Suppose that H is not right oscillatory at the point ξ and that $\Phi_n = \{\varphi_i\}_0^{n-1}$ is some right basis of ξ. Analogously [4-5], we may think of the quantity

$$r(x, \xi + 0) = \max\{k : x(t) = o(\varphi_{k-1}(t)), t \downarrow \xi\}$$

as the right multiplicity of a zero of the function $x(t) \in C_{[a,b]}$ at the point ξ; if the limit of $x(t)/\varphi_0(t)$ as $t \downarrow \xi$ exists and is nonzero, we set $r(x, \xi + 0) = 0$. The right multiplicity $r(x, \xi + 0)$ exists for every function $x(\cdot) \in H$. It is easily seen that the quantity $r(x, \xi + 0)$ is independent of which right basis of ξ is selected in its definition. The left multiplicity $r(x, \xi - 0)$ is defined analogously if, obviously, H is not left-oscillatory at the point ξ. If H is the space of polynomials of degree $n - 1$, $x(\cdot)$ a sufficiently smooth function and $r(x, \xi - 0) = r(x, \xi + 0) = \gamma$, then γ will coincide with the ordinary algebraic multiplicity of 0. In the present section we are concerned with the zeros and their multiplicity only for functions belonging to H.

Suppose that H is not oscillatory on some $I \subset [a, b]$. Then for any $x \in H$ the left or right multiplicity is defined at every point $\xi \in I$. We say a set I is directed if for every $\xi \in I$ only one basis may be selected, i.e., every point $\xi \in I$ may be a zero for $x(\cdot) \in H$ with only one direction, either left or right. We will say that H possesses the ET-property on a directed I if for any $x(\cdot) \in H$ the total multiplicity of its zeros on I does not exceed $n - 1$. The analogous concept for the case $I = [a, b]$ and $H \subset C_{[a,b]}^{n-1}$ was introduced in [6].

Suppose that the sets $\hat{\xi} = \{\xi_i\}_1^m \subset (a, b)$, $\hat{\nu} = \{\nu_i\}_1^m \subset \mathfrak{N} = \{0, \ldots, n - 1\}$ are defined in such a way that $\xi_1 < \xi_2 < \ldots < \xi_m$ and $\sigma(\hat{\nu}) = \nu_1 + \nu_2 + \ldots + \nu_m \le n$. Suppose that $F_n = \{f_i(t)\}_0^{n-1}$ is a basis in H and that $\{\varphi_i^k(t)\}_{i=0}^{n-1}$ is a basis of the point ξ_k, $k = \overline{1, m}$. Then for every k there exists a matrix $\|c_{ij}^k\|$, $i, j = \overline{0, n-1}$ such that

$$f_i(t) = c_{i0}^k \varphi_0^k(t) + \ldots + c_{in-1}^k \varphi_{n-1}^k(t), \quad i = \overline{0, n-1}.$$

If $\sigma(\hat{\nu}) = n$, we introduce the notation

$$D(\xi_1^{\nu_1}, \ldots, \xi_m^{\nu_m}) = \begin{vmatrix} c_{00}^1 & \ldots & c_{0\nu_1-1}^1 c_{00}^2 & \ldots & c_{0\nu_2-1}^2 & \ldots & c_{0\nu_m-1}^m \\ c_{10}^1 & \ldots & c_{1\nu_1-1}^1 c_{10}^2 & \ldots & c_{1\nu_2-1}^2 & \ldots & c_{1\nu_m-1}^m \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n-10}^1 & \ldots & c_{n-1\nu_1-1}^1 c_{n-10}^2 & \ldots & c_{n-1\nu_2-1}^2 & \ldots & c_{n-1\nu_m-1}^m \end{vmatrix}.$$

We will say that H possesses the W-property (is a W-space) if for every basis F_n of H, every set $\hat{\xi} = \{\xi_k\}_1^m \subset (a, b)$, and any basis $\{\varphi_i^k\}_{i=0}^{n-1}$ of the point ξ_k, $k = \overline{1, m}$, the corresponding determinant $D(\xi_1^{\nu_1}, \ldots, \xi_m^{\nu_m})$ is nonzero for any choice of the set $\hat{\nu} = \{\nu_i\}_1^m$ such that $\sigma(\hat{\nu}) = n$.

<u>THEOREM 2.</u> Suppose that H is nonoscillatory on the directed closed interval $[a, b]$ of the space and that $I \subset [a, b]$. Then the following assertions are equivalent: (a) if $\hat{\xi} \in I$, the dimension of the space

$$E(\hat{\xi}, \hat{\nu}) = \{x(t) \in H\colon\ r(x, \xi_i) \geqslant \nu_i,\ i = \overline{1, m}\}$$

satisfies the equality $\dim E(\hat{\xi}, \hat{\nu}) = n - \sigma(\hat{\nu}) = n - \nu_1 - \ldots - \nu_m$; (b) H is a generalized ET-space on I; (c) H possesses the W-property on I; (d) for any $\hat{\xi} \in I$ and $\nu_1 + \nu_2 + \ldots + \nu_m = n - 1$, there exists a unique normed function $x(t) \in H$ that satisfies the equality $r(x, \xi_i) = \nu_i$, $i = \overline{1, m}$.

<u>Proof.</u> If $x(t) \not\equiv 0$, $x \in H$, $x(t)$ will occur in some space $E(\hat{\xi}, \hat{\nu})$, moreover $\dim E(\hat{\xi}, \hat{\nu}) \geq 1$, since $E(\hat{\xi}, \hat{\nu}) \ni x(t)$ is nonempty. But $\dim E(\hat{\xi}, \hat{\nu}) = n - \sigma(\hat{\nu})$, consequently, $\sigma(\hat{\nu}) = n - \dim E(\hat{\xi}, \hat{\nu}) \leq n - 1$. That is, (a) → (b).

Let us show that if H is an ET-space on I, it possesses the W-property on I. We will use mathematical induction on the number of points in the set $\hat{\xi} = \{\xi_i\}_1^m$. Suppose $m = 1$, then $\nu_1 = n$ and

$$D(\xi_1^{\nu_1}) = \begin{vmatrix} c_{00}^1 & \ldots & c_{0n-1}^1 \\ \cdot & \cdot & \cdot \\ c_{n-10}^1 & \ldots & c_{n-1n-1}^1 \end{vmatrix}$$

is the determinant of the matrix connecting the two bases (the basis F_n and the basis $\{\varphi_i^1\}_0^{n-1}$ of the point ξ_1). Consequently, it is nonzero.

Let us now suppose that $D(\xi_1^{\nu_1}, \ldots, \xi_m^{\nu_m}) \neq 0$ for every $m \leq p$ for any set $\hat{\nu} = \{\nu_i\}_1^m \subset \mathfrak{N}$ such that $\sigma(\hat{\nu}) = n$. We will show that, in this case, for every $\mu \in \mathfrak{N}$

$$D(\xi_1^{\nu_1}, \ldots, \xi_p^{\nu_p}, \xi_{p+1}^{\mu}) \neq 0, \quad \nu_1 + \ldots + \nu_p = n - \mu.$$

If $\mu = 0$, this relation follows from the induction hypothesis. Let us assume that it is true for all $\mu \leq \mu_0 \leq n - 2$, and show that it holds for $\mu_0 + 1$.

Suppose that $B = \|b_{ij}\|_0^{n-1}$ is the matrix that connects the basis $\{\psi_i\}_0^{n-1}$ of the point ξ_{p+1} and the basis F_n. We introduce the function

$$x(t) = \begin{vmatrix} c_{00}^1 & \ldots & c_{0\nu_1-1}^1 c_{00}^2 & \ldots & c_{0\nu_p-1}^p b_{00} & \ldots & b_{0\mu_0-1} f_0(t) \\ c_{n-10}^1 & \ldots & c_{n-1\nu_1-1}^1 c_{n-10}^2 & \ldots & c_{n-1\nu_p-1}^p b_{n-10} & \ldots & b_{n-1\mu_0-1} f_{n-1}(t) \end{vmatrix}. \qquad (1)$$

Since F_n is connected to every basis $\{\varphi_i^k\}_{i=0}^{n-1}$, $k = \overline{1, p}$ corresponding to the matrix $\|c_{ij}^k\|$, for any $k = 1, \ldots, p$ in a neighborhood of the point ξ_k

$$x(t) = \gamma D(\xi_1^{\nu_1}, \ldots, \xi_{k-1}^{\nu_{k-1}}, \xi_k^{\nu_k+1}, \xi_{k+1}^{\nu_{k+1}}, \ldots, \xi_p^{\nu_p}, \xi_{p+1}^{\mu_0})\,\varphi_{\nu_k}^k(t) + o(\varphi_{\nu_k}^k(t))$$

(here $|\gamma| = 1$) as $t \to \xi_k$. By hypothesis,

$$D(\xi_1^{\nu_1}, \ldots, \xi_{k-1}^{\nu_{k-1}}, \xi_k^{\nu_k+1}, \xi_{k+1}^{\nu_{k+1}}, \ldots, \xi_p^{\nu_p}, \xi_{p+1}^{\mu_0}) \neq 0, \quad \begin{array}{l} \nu_1 + \ldots + \nu_p + 1 = n - \mu_0, \\ \nu_1 + \ldots + \nu_p = n - \mu_0 - 1. \end{array}$$

Consequently, $x(t) \not\equiv 0$ and $r(x, \xi_k) = \nu_k$, $k = \overline{1, p}$. For the point ξ_{p+1}, we may analogously write, using the matrix $\|b_{ij}\|$ that

$$x(t) = D(\xi_1^{\nu_1}, \ldots, \xi_p^{\nu_p}\xi_{p+1}^{\mu_0+1})\,\psi_{\mu_0}(t) + o(\psi_{\mu_0}(t)) \quad \text{as} \quad t \to \xi_{p+1}.$$

Hence it follows that $D(\xi_1^{\nu_1}, \ldots, \xi_p^{\nu_p}, \xi_{p+1}^{\mu_0+1} \neq 0$, since otherwise $r(x, \xi_{p+1}) \geq \mu_0 + 1$, which would lead to the relation

$$\sum_{k=1}^{p} r(x, \xi_k) + r(x, \xi_{p+1}) = \sum_{k=1}^{p} \nu_k + r(x, \xi_{p+1}) \geqslant n - \mu_0 - 1 + \mu_0 + 1 = n,$$

which contradicts the fact that H is an ET-space. Consequently,

$$D(\xi_1^{\nu_1}, \ldots, \xi_p^{\nu_p}, \xi_{p+1}^{\mu}) \neq 0, \quad \nu_1 + \ldots + \nu_p = n - \mu - 1, \quad \mu \leqslant n - 2.$$

This relation also holds for $\mu = n - 1$, since then $m = 1$. Thus, the determinant $D(\xi_1^{\nu_1}, \ldots, \xi_m^{\nu_m}) \neq 0$ also when $m = p + 1$, i.e., is nonzero for all $m = 1, \ldots, n - 1$. That is, (b) → (c).

That the implication (c) → (d) is valid may be proved by using the function (1) considered earlier:

$$x(t) = \begin{vmatrix} c_{00}^1 & \ldots & c_{0\nu_1-1}^1 c_{00}^2 & \ldots & c_{0\nu_m-1}^m f_0(t) \\ c_{n-10}^1 & \ldots & c_{n-1\nu_1-1}^1 c_{n-10}^2 & \ldots & c_{n-1\nu_m-1}^m f_{n-1}(t) \end{vmatrix},$$

where $F = \{f_i\}_0^{n-1}$ is a basis from H which is connected to every basis $\{\varphi_i^k\}_{i=0}^{n-1}$ of the points ξ_k, $k = \overline{1, m}$, by means of the matrix $\|c_{ij}^k\|_{i,j=0}^{n-1}$. As in the argument presented in the preceding section, it may be found that the function thus constructed satisfies the inequalities $r(x, \xi_k) = \nu^k$, i.e., it is the desired function. Consequently, (c) → (d).

To complete the proof, we must close the chain, i.e., prove that (d) → (a).

Suppose that (d) is false. That (a) holds when $m = 1$ is self-evident. Suppose that (a) holds when $m \leq p$; let us show that (a) holds when $m = p + 1$. Consider the spaces

$$\mathfrak{M} = \{x \in H : r(x, \xi_i) \geqslant \nu_i, \ i = \overline{1, p}\},$$
$$\mathfrak{M}_k = \{x \in \mathfrak{M} : r(x, \xi_{p+1}) \geqslant k\}.$$

From (d) it follows that there exists a basis $\{\psi_i(t)\}_0^{n-1}$ of ξ_{p+1}, such that $\psi_i \in \mathfrak{W}_i$ if $i \leq n - \sigma_p(\hat{\nu})$. Since ψ_i are linearly independent, and since $\dim \mathfrak{W} = n - \nu_1 - \ldots - \nu_p = k_p$ (by the induction hypothesis), the system $\{\psi_i\}_0^{k_p}$ is a basis in $\mathfrak{W}$. Moreover, for $\psi_i \in \mathfrak{W}_k$ it is necessary that $i \geq k$. Therefore, the system $\{\psi_i\}_k^{k_p}$, $k_p = n - \sigma_p(\hat{\nu})$ is a basis in $\mathfrak{W}$. But this means that $\dim \mathfrak{W}_k = k_p - k = n - 1 - \nu_1 - \ldots - \nu_p - k$ for any $k \leq k_p$. Consequently, (d) → (a). The theorem is proved.

3. Let us assume that $H \subset C_{[a,b]}$ is a $T_{(n-1)}$-space on [a, b] and that it is regular at every point of [a, b]. Under these assumptions, the following theorem, the main result of the article holds.

THEOREM 3. Suppose that H is a regular T_{n-1}-space on [a, b). Then

(a) for any $x(t)\in H$ $\sum_{\xi\in[a,b)} r(x,\xi)\leqslant n-1$ i.e., H is a generalized ET-space on [a, b];

(b) suppose that natural numbers $r_1, \ldots, r_m$ are defined so that $r_1 + \ldots + r_m = n - 1$. Then for any set $\xi_1 < \xi_2 < \ldots < \xi_m$ in [a, b) there exists a unique normed function $x(t) \in H$ such that $r(x, \xi_i) = r_i$, $i = \overline{1, m}$;

(c) suppose that the sets $\hat{\xi} = \{\xi_i\}_1^m$ and $\hat{r} = \{r_i\}_1^m$ are defined so that

$$a\leqslant \xi_1<\ldots<\xi_m\leqslant b,\quad \sigma(\hat{r})=r_1+\ldots+r_m\leqslant n;$$

then the dimension of the space

$$E(\hat{\xi},\hat{r})=\{x(t)\in H: r(x,\xi_i)\geqslant r_i,\ i=\overline{1,m}\}$$

is equal to $n - \sigma(\hat{r})$;

(d) H possesses the W-property on [a, b).

As a consequence of the preceding theorem, properties (a)-(d) are equivalent. Consequently, it suffices to prove any one of them. Let us prove that (c) is valid. For this purpose we will need the following assertions.

LEMMA 1 [6]. Suppose that F is a T-system on (a, b) and that ξ and η are arbitrary ($\xi < \eta$) points in [a, b]. If one of these points is regular, F will not be oscillatory on the pair $\{\xi, \eta\}$ (there exists such a right basis of ξ that, enumerated in reverse order, will be a left basis of η). Then the corresponding system $\{z_i\}_0^{n-1}$ H(F) which satisfies the equalities $r(z_i, \xi) = i = n - 1 - r(z_i, \eta)$, $i = \overline{0, n}$ possesses the following property: For any, $k \leq m$ ($\leq n - 1$), the functions $\{z_i\}_k^n$ form a T_{-k}^n-system on $(a, b)\backslash\{\xi, \eta\}$.

LEMMA 2. Suppose that $F \subset C_{[a,b]}$ is a T_n-system on the set $\Omega = (a, a_1) \cup (a_1, a_2) \cup \ldots \cup (a_m, b)$, where $a < a_1 < \ldots < a_m < b$. Suppose ξ is some point of Ω and that $\Phi=\{\varphi_i\}_0^{n-1}$ is an arbitrary basis of this point (on Ω). Then F is an M-system on $\Omega\backslash\{\xi\}$, i.e., for every k the system $\{\varphi_i\}_{n-1-k}^{n-1}$ is a T_k-system on $\Omega\backslash\{\xi\}$.

Proof. Suppose that $\Phi=\{\varphi_i\}_0^{n-1}$ is a left basis of the point $\xi \in [a, b]$. To prove the lemma, it is sufficient to prove that for every k = 0, 1, ..., n − 1, the determinant

$$\Delta[\varphi_k(t_k),\ldots,\varphi_{n-1}(t_{n-1})]=\det\|\varphi_i(t_j)\|_k^{n-1}=$$
$$=\begin{Vmatrix}\varphi_k(t_k) & \varphi_k(t_{k+1}) \ldots \varphi_k(t_{n-1})\\ \varphi_{k+1}(t_k) & \varphi_{k+1}(t_{k+1}) \ldots \varphi_{k+1}(t_{n-1})\\ \cdots & \cdots\\ \varphi_{n-1}(t_k) & \varphi_{n-1}(t_{k+1}) \ldots \varphi_{n-1}(t_{n-1})\end{Vmatrix}$$

is nonzero for any $t_k, \ldots, t_{n-1}$ (all distinct) in $(a, \xi) \cup (\xi, b)$.

Let us first show that for every k, this determinant retains its sign for any choice of points $t_k < t_{k+1} < \ldots < t_{n-1}$ in $(a, \xi) \cup (\xi, b)$ if the number of these points in (a, ξ) is fixed. Suppose that (a, ξ) contains $p - k$ points $t_k, t_{k+1}, \ldots, t_p$, $k \leq p \leq n$. We consider the points $\tau_0 < \tau_1 < \ldots < \tau_{k-1} < \xi$ in a neighborhood of ξ that do not contain zeros for all φ_i such that $\tau_i \in (t_p, \xi)$.

From the properties of the determinant and the definition of the basis $(\varphi_i(t)=o(\varphi_j(t))$ it follows that, as $t \uparrow \xi$ where i > j,

$$\delta_0=\begin{vmatrix}\varphi_0(t_k) & \cdots \varphi_0(t_p)\,\varphi_0(\tau_0) & \cdots \varphi_0(\tau_{k-1}) & \cdots \varphi_0(t_{n-1})\\ \varphi_{n-1}(t_k) & \cdots \varphi_{n-1}(t_p)\,\varphi_{n-1}(\tau_0) & \cdots \varphi_{n-1}(\tau_{k-1}) & \cdots \varphi_{n-1}(t_{n-1})\end{vmatrix}=$$
$$=(-1)^{(p-k)(k-1)}\begin{vmatrix}\varphi_0(\tau_0) & \cdots\varphi_0(\tau_{k-1})\,\varphi_0(t_k) & \cdots\varphi_0(t_p) & \cdots\varphi_0(t_{n-1})\\ \varphi_{n-1}(\tau_0)\cdots & \varphi_{n-1}(\tau_{k-1})\,\varphi_{n-1}(t_k)\cdots & \varphi_{n-1}(t_p)\cdots & \varphi_{n-1}(t_{n-1})\end{vmatrix}=$$

$$= (-1)^{(n-k)(k-1)} \varphi_0(\tau_0)[\ldots[\varphi_{k-1}(\tau_{k-1}) \det \| \varphi_i(t_i) \|_k^{n-1} + o(\varphi_{k-1}(\tau_{k-1}))] \ldots] + o(\varphi_0(\tau_0)).$$

Moreover, by the T-property of Φ the determinant δ_0 does not change sign (more precisely, does not vanish) for τ_i close enough to ξ. Consequently, the sign of the determinant $\det \| \varphi_i(t_i) \|_k^{n-1}$ does not depend on the choice of the points $t_k < \ldots < t_{n-1}$ in $(a, \xi) \cup (\xi, b)$, if the number of such points in (a, ξ) is fixed.

Let us now prove that as a result and from the fact that $\det \| \varphi_i(t_i) \|_0^{n-1} \neq 0$ which holds for any set of points $t_0 < t_1 < \ldots < t_{n-1}$ in $(a, \xi) \cup (\xi, b)$, it follows that $\det \| \varphi_i(t_i) \|_k^{n-1} \neq 0$ for any k for all these points.

Let us first establish this fact for $k = n - 1$. We assume the contrary, supposing that $\varphi_{n-1}(t_{n-1}) = 0$ for $t_{n-1} \in (a, \xi) \cup (\xi, b)$. Let us select points t_{n-2}^1 and t_{n-2}^2 so that $a < t_{n-2}^1 < t_{n-1} < t_{n-2}^2 < \xi$ and $\varphi(t_{n-2}^1)\varphi(t_{n-2}^2) \neq 0$. Then the numbers

$$\begin{vmatrix} \varphi_{n-2}(t_{n-2}^1) & \varphi_{n-2}(t_{n-1}) \\ \varphi_{n-1}(t_{n-2}^1) & \varphi_{n-1}(t_{n-1}) \end{vmatrix} = -\varphi_{n-2}(t_{n-1})\varphi_{n-1}(t_{n-2}^1)$$

and

$$\begin{vmatrix} \varphi_{n-2}(t_{n-1}) & \varphi_{n-2}(t_{n-2}^2) \\ \varphi_{n-1}(t_{n-1}) & \varphi_{n-1}(t_{n-2}^2) \end{vmatrix} = \varphi_{n-2}(t_{n-1})\varphi_{n-1}(t_{n-2}^2)$$

cannot have different signs by virtue of what we have already proved, which in the case $\varphi_{n-2}(t_{n-1}) \neq 0$ contradicts the fact that $\varphi_{n-1}(t)$ has a constant sign on both sides of the point ξ, while the points t_{n-2}^1 and t_{n-2}^2 are on the same side of ξ. Consequently, $\varphi_{n-2}(t_{n-1}) = 0$.

Analogously, it may be proved that if $\varphi_{n-1}(t_{n-1}) = 0$ then $\varphi_k(t_{n-1}) = 0$ for all $k = 0, \ldots, n$ though this is impossible by virtue of the T-property of Φ. By the contradiction just obtained, it may be declared that $\varphi_{n-1}(t_{n-1}) \neq 0$.

Next, let us introduce the functions

$$\psi_i^k(t) = \Delta[\varphi_i(t_k), \varphi_k(t_{k+1}), \ldots, \varphi_{n-2}(t_{n-1}), \quad \varphi_{n-1}(t)], \quad 0 \leqslant i < k,$$

defined by the number $k(\leq n - 1)$ and a fixed set of points $t_k < \ldots < t_{n-1}$ belonging to $(a, \xi) \cup (\xi, b)$.

By Sylvester's determinant identity [7, p. 48], for every k and any set $\{t_i\}_0^{n-1}$ belonging to $(a, \xi) \cup (\xi, b)$,

$$\det \| \psi_i^k(t_j) \|_{i,j=\nu}^{k-1} = [\det \| \varphi_i(t_j) \|_k^{n-1}]^{n-2-\nu} \det \| \varphi_i(t_j) \|_\nu^{n-1}, \tag{2}$$

where $\nu = 0, 1, \ldots, k - 1$. By what we proved above, $\varphi_{n-1}(t)$ does not have any zeros in $(a, \xi) \cup (\xi, \beta)$. Therefore, for $\nu = 0$ it follows from (2) that the system $\{\psi_i^{n-1}\}_{i=0}^{n-2}$ posesses a T-property on $(a, \xi) \cup (\xi, b)$. Moreover, by (2) for any ν the determinant $\Delta[\psi_\nu^{n-1}(t_\nu), \ldots, \psi_{n-2}^{n-1}(t_{n-2})]$ does not change side when the points $t_\nu < t_{\nu+1} < \ldots < t_{n-2}$ in $(a, \xi) \cup (\xi, b)$ are varied if the number of these points to the left of ξ is fixed. Therefore, the conditions of the preceding arguments may be imposed on the system $\{\psi_i^{n-1}(t)\}_0^{n-2}$, whence it follows that $\psi_{n-2}^{n-1}(t) = \Delta[\varphi_{n-2}(t_{n-1}), \varphi_{n-1}(t)]$ does not assume zero values at points in (a, b) other than ξ and t_{n-1}. Carrying out analogous arguments for families or functions $\{\psi_i^{k+1}(t)\}_{i=0}^k$ in succession, we ultimately find that $\psi_k^{k+1}(t) = (-1)^{n-1-k}\Delta[\varphi_k(t_k), \ldots, \varphi_{n-1}(t_{n-1})] \neq 0$ for any k. The lemma is proved.

Proof of Theorem 3. By Theorem 2, it suffices to prove that property (c) holds. For this purpose let us first show that if $F = \{f_i\}_0^{n-1}$ is some basis of the point $\xi \in [a, b)$ and $F^k = \{f_i\}_k^{n-1}$, for any other point $\eta \in (a, b)$ and any function $x(t) \in H(F^k)$, its F^k-multiplicity at the point η coincides with the H-multiplicity at this point. In fact, suppose that $\Phi = \{\varphi_i\}_0^{n-1}$ is a basis of the point ξ such that $v_i(t) \equiv \varphi_{n-1-i}(t)$, $i = \overline{0, n-1}$, forming a basis of the point η (by Lemma 1, such a basis rcidyd for a regular space) and $\Phi^k = \{\varphi_i\}_k^{n-1}$ then $H(\Phi k) = H(Fk)$ and the system $V_k = \{v_i\}_0^{n-1-k}$ is a basis of η relative to $H(\Phi k)$. Therefore, if the Φk-multiplicity of $x(t) \in H(\Phi)$ at the point η is equal to m, for certain $\alpha_m, \alpha_{m+1}, \ldots, \alpha_{n-1-k}$ we must have

$$x(t) = \alpha_m v_m(t) + \ldots + \alpha_{n-1-k} v_{n-1-k}(t), \ \alpha_m \neq 0.$$

But this means that the H-multiplicity of x(t) at the point η is equal to m.

Let us turn to the proof of (c). Suppose that $a \leq \xi_1 < \ldots < \xi_m < b$ and $\hat{\nu} = \{\nu_i\}_1^m$ is a set of natural numbers such that $\sigma(\hat{\nu}) = \nu_1 + \ldots + \nu_m \leq n$.

By Lemma 2, the set $E^1 = \{x \in E_n : r(x, \xi_1) \geq \nu_1\}$ is a regular $T_{(n-1-\nu_1)}$-space on (ξ_1, b) and by virtue of the preceding arguments, for every function $x(t) \in E^1$ its E^1-multiplicity at the point ξ_2 coincides with the H-multiplicity of x(t) at this point. Therefore, by Lemma 2, applied to the space E^1 at the point ξ_2, the set

$$E^2 = \{x \in E^1 : r(x, \xi_2) \geqslant \nu_2\} = \{x \in H : r(x, \xi_1) \geqslant \nu_1, \quad r(x, \xi_2) \geqslant \nu_2\}$$

is a $T_{(n-1-\nu_1-\nu_2)}$-space on (ξ_2, b). Continuing the argument, we find that the set $E^m = \{x \in E^{m-1} : r(x, \xi_m) \geq \nu_m\} = E(\hat{\xi}, \hat{\nu})$ is a $T_{(n-1-\sigma(\hat{\nu}))}$-space on (ξ_n, b) and has, consequently, dimension $n - \sigma(\hat{\nu})$. The theorem is proved.

4. A continuous function $\varphi(t)$ will be said to be a T_{n-1}-continuation of a T-system F on Ω, if the augmented system $F \cup \{\varphi\}$ is also a T-system on Ω.

<u>THEOREM 4.</u> Suppose that F is a regular T-system on [a, b] and that $\varphi(t)$ is a regular T-continuation of F, i.e., the augmented system $F \cup \{\varphi\}$ is a regular T_n-system on [a, b). Then the total F-multiplicity of the zeros of φ on [a, b) does not exceed n.

<u>Proof.</u> We let $\hat{F}$ denote a regular Chebyshev system $f_0, \ldots, f_{n-1}, \varphi$. By the preceding theorem, for any $x(t) \in H(\hat{F})$, the total $\hat{F}$-multiplicity of the zeros of x(t) on [a, b) does not exceed the order of $\hat{F}$ (i.e., n). Therefore, we need only prove that for every $x(t) \in H(\hat{F})$ and any point $\xi \in [a, b)$, the F-multiplicity of the latter point $r(x, \xi)$ does not exceed the $\hat{F}$-multiplicity $\hat{r}(x, \xi)$. Suppose that $G = \{g_i\}_0^{n-1}$ and $\Psi = \{\psi_i\}_0^n$ are bases of the point ξ relative to the T-systems F and $\hat{F}$, respectively. Since $G \cup \varphi$ is a basis in $H(\hat{F})$, for certain $\{\alpha_{ij}\}_{i,j=0}^{n-1}$ and $\{\beta_i\}_0^n$,

$$\psi_i(t) = \sum_{j=0}^{n-1} \alpha_{ij} g_j(t) + \beta_i \varphi(t), \quad i = \overline{0, n}. \tag{3}$$

Hence, it follows that if $r(\varphi, \xi) \neq 0$ (i.e., $\varphi(\xi) = 0$), then $\alpha_{00} \neq 0$. Obviously, $\alpha_{i0} = 0$ if $i > 0$, since $\psi_i(t) = o(\psi_0(t))$ as $t \to \xi$ when $i > 0$.

Let us show that $r_0 = r(x, \xi)$. If $r_0 > 1$, we divide (3) by $g_1(t)$ and take limits as $t \to \xi$. We find that $\psi_1(\xi)/g_1(\xi) = \alpha_{11}$ (since $g_i = o(g_1(t))$ if $i > 1$). Consequently, $\psi_1(t)_{t \to \xi} = O(g_1(t))$ and then $\alpha_{i1} = 0$ if $i > 1$. Carrying out an analogous operation successively for all $k = 0, \ldots, r_0 - 1$, we find that

$$\psi_k(t) = O(g_k(t)), \quad t \to \xi, \quad k = 0, \ldots, r_0 - 1.$$

Hence, and also from the definition of multiplicity, it follows that $\hat{r}(x, \xi) \geq r_0$. The theorem is proved.

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DUAL EQUATIONS OF CONVOLUTION TYPE WITH KERNELS FROM DIFFERENT BANACH ALGEBRAS

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We study dual integral equations of convolution type with kernels generated by functions from different Banach algebras of the type $L_1(-\infty, \infty)$ with weights, and defined by an operator equation. We establish theorems on solvability and Fredholmness, representations of solutions and of the resolvent kernel, and formulas for calculating the characteristic and the index of the corresponding operator.

A number of problems, including, in particular, some in mathematical and theoretical physics, astrophysics, elasticity theory, waveguide theory, and exploratory geophysics reduce to the solution of integral equations with kernels which depend on the difference of arguments. A brief history of the study of such equations of convolution type as the Wiener–Hopf, dual integral equations, and the corresponding conjugate equations can be found in the works of F. D. Gakhov, M. G. Krein, V. I. Smagina, V. I. Smirnov, G. N. Chebotarev, Yu. I. Cherskii, and others.

It is noted in [1] that dual integral equations of the form

$$\begin{aligned} &\varphi(t) - \int_{-\infty}^{\infty} k_1(t-s)\varphi(s)\,ds = f(t), \quad -\infty < t < 0, \\ &\varphi(t) - \int_{-\infty}^{\infty} k_2(t-s)\varphi(s)\,ds = f(t), \quad 0 < t < \infty, \end{aligned} \tag{1}$$

arise, for example, in the determination of the potential of an electrified disk.

In what follows, we will consider (1) with regard to the unknown function $\varphi(t)$ and the operator induced by (1), assuming that, for some constant $c > 0$, $k_1(t)$, $k_2(t)\exp(ct) \in L_1(-\infty, \infty)$ $(\equiv L)$.

Using the customary notation we can write (1) in the form

$$p^-\{\varphi*[\delta - k_1]\}(t) = f^-(t), \quad p^+\{\varphi*[\delta - k_2]\}(t) = f^+(t), \quad -\infty < t < \infty. \tag{2}$$

We note that in case $k_1(t)$, $k_2(t)\exp(ct) \in L$ it is easy to deduce that $k_j(t)\exp(c_j t) \in L$, $j = 1, 2$. But if $c < 0$, then difficulties arise similar to those mentioned in [1-3]. This case has not yet been studied without substantial additional restrictions.

Just as for integral equations of Wiener–Hopf type [4] obtained from (1) with $k_1(t) = 0$, $-\infty < t < \infty$; $f(t) = 0$, $t < 0$, the theory of equations of the form (1) turns out to be much more complicated than that for the simplest equations of convolution type

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