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ON THE BEHAVIOR OF SOLUTIONS OF OPERATOR-DIFFERENTIAL EQUATIONS AT INFINITY

ПРО ПОВЕДІНКУ НА НЕСКІНЧЕННОСТІ РОЗВ'ЯЗКІВ ДИФЕРЕНЦІАЛЬНО-ОПЕРАТОРНИХ РІВНЯНЬ

The existence of limits at the infinity, generalized in the Abel sense, is established for bounded solutions of the operator-differential equation $y'(t) = Ay(t)$ in a reflexive Banach space.

У рефлексивному банаховому просторі встановлено існування узагальнених у розумінні Абеля границь на нескінченності для обмежених розв'язків операторно-диференціального рівняння $y'(t) = Ay(t)$.

We consider a Cauchy problem

$$y'(t) = Ay(t), \quad t \in \mathbb{R}_+, \quad y(0) = y_0, \tag{1}$$

in a Banach space $\mathfrak{B}$ endowed with the norm $\|\cdot\|$. Here, A is a linear closed operator in $\mathfrak{B}$ and $\mathbb{R}_+ = [0, \infty)$. A function $y(t)$ is said to be a solution of the Cauchy problem (1) if it satisfies both equalities in (1) and $y(t) \in C^1(\mathbb{R}_+, \mathfrak{B})$.

In the present paper, we are concerned with the behavior of solutions of the Cauchy problem (1) at the infinity.

Definition. *Let* $\alpha > 0$ *and let* $y(t) \in C(\mathbb{R}_+, \mathfrak{B})$. *We define the Cesaro limit of* $y(t)$ *of order* α *as*

$$(C, \alpha) \lim_{t \to \infty} y(t) = \lim_{t \to \infty} \alpha t^{-\alpha} \int_0^t (t-s)^{\alpha-1} y(s)\, ds,$$

whenever the latter exists.

Theorem 1 [1]. *Let* A *be a generator of a strongly continuous semigroup* $T(t)$, $t \in \mathbb{R}_+$. *Then*

a) *if* $x = x_0 + x_1 \in N(A) \oplus \overline{R(A)}$, *then* $(C, \alpha) \lim\limits_{t \to \infty} T(t)x = x_0$;

b) *if there exists a sequence* $\{t_j, j \in \mathbb{N}\}$, $t_j \to \infty$, *such that sequence*

$$\alpha t_j^{-\alpha} \int_0^{t_j} (t_j - s)^{\alpha-1} y(s)\, ds$$

is weakly convergent, then $x \in N(A) \oplus \overline{R(A)}$;

c) *if* $\mathfrak{B}$ *is a reflexive space, then* $\mathfrak{B} = N(A) \oplus \overline{R(A)}$ *and the limit*

$$(C, \alpha) \lim_{t \to \infty} T(t)x$$

exists $\forall\, x \in \mathfrak{B}$.

Let $y(t)$ be a bounded solution of the Cauchy problem (1). Then statement a) of Theorem 1, generally speaking, is not true. It is shown by the following example:

Example. We consider a space $\mathfrak{M}$ of all bounded sequences $\{\beta_n \in \mathbb{C}, n \in \mathbb{N} \cup \{0\}\}$ equipped with the norm $\|\{\beta_n\}\| = \sup |\beta_n|$. We set $A\{\beta_n\} = \{\gamma_n\}$, where $\gamma_0 = 0$, $\gamma_1 = \beta_0$, $\gamma_n = i\beta_n/n + \beta_0$, $n \geq 2$. Let $\mathfrak{M}_0 = \left\{\{\beta_n\} \in \mathfrak{M}, \beta_0 = 0\right\}$. The restriction of A to $\mathfrak{M}_0$ (we denote it by A_0) generates a C_0-semigroup of contractions $T(t)$ [2, p. 535], and the vector $\{0, 1, 1,$

$1, \dots \} \overline{\in} N(A_0) \oplus \overline{R(A_0)}$. We conclude from Theorem 1 that the (C, α)-limit of a bounded solution of the Cauchy problem (1), where $y(0) = \{0, 1, 1, 1, \dots\}$, does not exist. But $y(0) = A\{1, 0, 0, \dots\}$, concluding the example. Also it is shown at statement a) of Theorem 1 is not valid for bounded solutions of the Cauchy problem (1) when A generates an unbounded C_0-semigroup.

Lemma 1. *Let $y(t)$ be a bounded solution of the Cauchy problem* (1). *Then statement* b) *of Theorem* 1 *holds true if we substitute $y(t)$ for $T(s)x$ and $y(0)$ for x.*

Proof. Since A is closed, we conclude that

$$\alpha t^{-\alpha} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \in D(A).$$

By letting $t \to \infty$, we get

$$A\left(\alpha t^{-\alpha} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \right) =$$

$$= \alpha t^{-\alpha} \int_0^{t-1} (t-s)^{\alpha-1} y'(s)\, ds + \alpha t^{-\alpha} \int_{t-1}^t (t-s)^{\alpha-1} y'(s)\, ds =$$

$$= \alpha t^{-\alpha} \int_{t-1}^t (t-s)^{\alpha-1} y'(s)\, ds + \alpha t^{-\alpha} y(t-1) - \alpha t^{-1} y(0) +$$

$$+ \alpha(\alpha-1) t^{-\alpha} \int_0^{t-1} (t-s)^{\alpha-2} y(s)\, ds \to 0$$

since $\| y(t) \|$ is bounded. Since A is closed, we obtain $x_0 \in N(A)$.

We set $z(t) = y(t) - x_0$. Then

$$\alpha t_j^{-\alpha} \int_0^{t_j} (t_j - s)^{\alpha-1} z(s)\, ds \overset{w}{\to} 0, \quad j \to \infty,$$

(here, $\overset{w}{\to}$ stands for the weak convergence in $\mathfrak{B}$). Integrating by parts, we get

$$z(0) = -A\left(t_j^{1-\alpha} \int_0^{t_j - 1} (t_j - s)^{\alpha-1} z(s)\, ds \right) + z(t-1) t^{1-\alpha} -$$

$$- (\alpha - 1) t_j^{1-\alpha} \int_0^{t_j - 1} (t_j - s)^{\alpha-2} z(s)\, ds.$$

When $j \to \infty$, the last two terms on the right-hand side of the above equality tend weakly to zero. Hence, $z(0) \in \overline{R(A)}$ and $y(0) \in N(A) \oplus \overline{R(A)}$.

Now we are going to generalize statement b) of Theorem 1.

Theorem 2. *Let $\mathfrak{B}$ be a reflexive Banach space. We suppose that the Cauchy problem* (1) *admits at most one bounded solution for any $y_0 \in \mathfrak{B}$ (i.e., if there exist few solutions for certain y_0, only one of them is bounded). If $y(t)$ is a solution of the Cauchy problem* (1) *such that $\| y(t) \| \le M$, then $\forall\, \alpha > 0$ there exists*

$$(C, \alpha) \lim_{t \to \infty} y(t) = z, \quad z \in N(A).$$

Proof. We denote by $\mathfrak{N}'$ the set of all $w \in \mathfrak{B}$ such that there exists a bounded solution of Cauchy problem (1) with the initial value w. For any $w \in \mathfrak{N}'$, we set $\| w \|_{\mathfrak{N}} = \sup \{ \| x(t) \|, \, t \geq 0 \}$, where $x(t)$ is the bounded solution of the Cauchy problem (1) corresponding to w by the definition of $\mathfrak{N}'$. We denote by $\mathfrak{N}$ the completion of $\mathfrak{N}'$ in the norm $\| \cdot \|_{\mathfrak{N}}$. We outline that $\forall \, w \in \mathfrak{N} \;\; \| w \|_{\mathfrak{N}} \geq \| w \|$.

Without loss of generality, we assume that $\mathfrak{N}$ is dense in $\mathfrak{B}$. If this is not the case, we consider the Cauchy problem (1) in the space $\mathfrak{B}_0 := \overline{\mathfrak{N}}$ (the bar denotes the closure in $\mathfrak{B}$). In $\mathfrak{B}_0$, all the assumptions of Theorem 2 hold. So, by using the continuity and denseness of the embedding $\mathfrak{N} \subset \mathfrak{B}$, we get $\mathfrak{B}^* \subset \mathfrak{N}^*$ with the continuous embedding.

We define a semigroup of operators $T(t)$, $t \geq 0$, on $\mathfrak{N}'$ by the relation $T(t) w = x(t)$, $t \geq 0$, where $x(t)$ is the solution of the Cauchy problem (1) corresponding to w by the definition of $\mathfrak{N}'$. It is easy to see that $T(t)$ is a semigroup of contractions; this is why $T(t)$ may be extended on $\mathfrak{N}$ by continuity.

We state that $T(t)$ is a C_0-semigroup on $\mathfrak{N}$. To prove this, it is sufficient to show that $T(t)$ is weakly continuous at zero ([3], IX.1). The latter condition holds if the functions $T(t)w$ are weakly continuous at zero $\forall \, w \in \mathfrak{N}'$. Otherwise,

$$\exists \; y_0^* \in \mathfrak{N}^* \;\; \exists \; \varepsilon > 0 \;\; \exists \; y_1 \in \mathfrak{N}' \;\; \exists \, \{ t_n, \, n \in \mathbb{N} \} \;\; (t_n \to 0, \; n \to \infty)$$

$$\left| \; y_0^*(T(t_n) y_1 - y_1) \; \right| > \varepsilon. \tag{2}$$

Obviously, y_0^* does not belong to the closure of $\mathfrak{B}^*$ in $\mathfrak{N}^*$.

Now we are going to make some preliminary constructions. Given any $z^* \in \mathfrak{N}^*$, we define the Banach space $\mathfrak{X} := (w^* + \alpha z^*, \; w^* \in \mathfrak{B}^*, \; \alpha \in \mathbb{C})$ with the norm $\| w^* + \alpha z^* \|_{\mathfrak{X}} = \| w^* \|_{\mathfrak{B}^*} + |\alpha|$. Since $\mathfrak{B}^*$ and $\mathfrak{X} / \mathfrak{B}^*$ are reflexive, $\mathfrak{X}$ is reflexive, too. We consider the function $T(t) y_1$ bounded in $\mathfrak{X}$. There exist a subsequence $\{ s_n, \, n \in \mathbb{N} \}$ of the sequence $\{ t_n, \, n \in \mathbb{N} \}$ and $w_0 \in \mathfrak{X}$ such that $T(s_n) y_1 \overset{w}{\to} w_0$ in $\mathfrak{X}$ as $n \to \infty$. Since $\mathfrak{B}$ is weakly closed and $\mathfrak{B}^*$ is contained in $\mathfrak{X}$, we conclude that $w_0 \in \mathfrak{B}$. With $T(t) y_1$ being strongly continuous in $\mathfrak{B}$, we see that $w_0 = y_1$. So, we get $y_0^*(T(s_n) y_1 - y_1) \to 0$, $n \to \infty$. It makes a contradiction to (2). Therefore, $T(t)$ is a C_0-semigroup on $\mathfrak{N}$.

Let us prove that there exists a sequence $\{ r_n, \, n \in \mathbb{N} \}$ $(r_n \to 0, \; n \to \infty)$ such that

$$T(r_n) y_0 \overset{w}{\to} w \;\; \text{in} \;\; \mathfrak{N}, \;\; n \to \infty. \tag{3}$$

Assume the contrary. We set $t_n = n$. The reflexivity of $\mathfrak{B}$ implies the existence of a subsequence $\{ s_n, \, n \in \mathbb{N} \}$ of $\{ t_n \}$ and $u \in \mathfrak{B}$ such that $T(s_n) y_0 \overset{w}{\to} u$ in $\mathfrak{B}$. Then there exist $y_0^* \in \mathfrak{N}^*$, a subsequence $\{ r_n, \, n \in \mathbb{N} \}$ of $\{ s_n \}$, and $\varepsilon > 0$ such that

$$\left| \; y_0^*(T(r_n) y_0 - u) \; \right| > \varepsilon. \tag{4}$$

Obviously, y_0^* does not belong to the closure of $\mathfrak{B}^*$ in $\mathfrak{N}^*$. We define the space $\mathfrak{X}$ in the same way as it was done after relation (2). Repeating this argument, we arrive at the conclusion that there exist a subsequence $\{ p_n, \, n \in \mathbb{N} \}$ of the sequence

$\{r_n\}$ and $w \in \mathfrak{B}$ satisfying the relation $y_0^*(T(r_n)y_0) \xrightarrow{w} u$ in $\mathfrak{X}$, $n \to \infty$. The latter condition makes a contradiction to (4). Therefore, (4) is not true and $\exists\, u \in \mathfrak{N}$ such that $T(r_n)y_0 \xrightarrow{w} u$ in $\mathfrak{N}$, $n \to \infty$ (we point out that $u \in \mathfrak{N}$ because $u \in N(A)$ by Lemma 1).

In view of (3), we need only to apply Theorem 1 to the semigroup $T(t)$ and y_0. From part b) of the theorem, we deduce that $y_0 \in N(B) \oplus \overline{R(B)}$ (B is a generator of $T(t)$). Part a) states that there exists

$$(C, \alpha) \lim_{t \to \infty} T(t)y_0 = u.$$

Thus, Theorem 2 is proved.

Corollary 1. *If the open right-side halfplane* $\{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda > 0\}$ *is not contained in the point spectrum* $\sigma_p(A)$, *then the statement of Theorem 2 holds true.*

Proof is immediately obtained from the proof of Theorem 23.7.1 [2].

Theorem 3. *Let* $\mathfrak{B}$ *be a reflexive Banach space. We suppose that* $\exists\, \lambda_1, \lambda_2 \in \mathbb{C}$, $(\mathrm{Re}\,\lambda_1 > 0,\ \mathrm{Re}\,\lambda_2 > 0)$ *such that there exist projection operators* P_1 *and* P_2 *onto the subspaces* $N_1 = \{x \in \mathfrak{B},\ Ax = \lambda_1 x\}$, $N_2 = \{x \in \mathfrak{B},\ Ax > \lambda_2 x\}$ *respectively. If* $y(t)$ *is a solution of the Cauchy problem* (1) *such that* $\|y(t)\| \le M$, *then* $\forall\, \alpha > 0$ *there exists*

$$(C, \alpha) \lim_{t \to \infty} y(t) = z,$$

and $z \in N(A)$.

Proof. We set $P_3 = I - P_2 - P_1$; $N_3 = P_3\mathfrak{B}$. We denote $y_i(t) := P_i y(t)$, $i \in 1, 2, 3$. By applying the operator $P_2 + P_3$ to (1), we get

$$y_2'(t) + y_3'(t) = (P_2 + P_3)A(y_2(t) + y_3(t)).$$

This is why the function $y_2(t) + y_3(t)$ is a bounded solution of the equation $z'(t) = (P_2 + P_3)Az(t)$, $t \ge 0$.

Since $\lambda_1 \overline{\in} \sigma_p((P_2 + P_3)A)$, we may apply Corollary 1 to the present setting. So,

$$\exists\ (C, \alpha) \lim_{t \to \infty} (y_2(t) + y_3(t)) = w, \quad w \in N((P_2 + P_3)A). \tag{5}$$

Here, $w \in N(A)$ because

$$\alpha t^{-\alpha} \int_0^t (t-s)^{\alpha-1} y_i(s)\, ds \in N_i, \qquad i = 1, 2, 3,$$

and $N((P_2 + P_3)A) = N(A) + N_1$.

In a similar way, we can obtain

$$\exists\ (C, \alpha) \lim_{t \to \infty} y_3(t) = v, \quad v \in N(P_3 A), \tag{6}$$

$$\exists\ (C, \alpha) \lim_{t \to \infty} (y_1(t) + y_3(t)) = u, \quad u \in N((P_1 + P_3)A) \tag{7}$$

by applying to (1) the operators P_3 and $P_1 + P_3$, respectively. From (5), (6), and (7), we deduce the statement of Theorem 3.

Corollary 2. *Let* $\mathfrak{B}$ *be a reflexive Banach space. If there exist* $\lambda \in \mathbb{C}$ $(\operatorname{Re}\lambda > 0)$ *and a projection operator* P *onto the subspaces* $\mathfrak{X} = \{x \in \mathfrak{B}, Ax = \lambda x\}$ *such that* A *is invariant on* $(I - P)\mathfrak{B}$, *then the statement of Theorem* 3 *holds true.*

Corollary 3. *If* $\mathfrak{B}$ *is a Hilbert space, then the statement of Theorem* 3 *remains valid.*

Corollary 4. *If there exist* $\lambda_1, \lambda_2 \in \mathbb{C}$, $\operatorname{Re}\lambda_1 > 0$, $\operatorname{Re}\lambda_2 > 0$, *such that the subspaces* N_1, N_2, *defined in Theorem* 3 *are finite-dimensional, then the statement of Theorem* 3 *holds true.*

When A satisfies some additional assumptions, we can reformulate Theorem 1 in a more precise way:

Theorem 4 [1]. *If* A *is a generator of a bounded analytic semigroup, then we can replace* (C, α)*-limits in Theorem* 1 *by the strong ones.*

Theorem 5. *Let the assumptions of Theorems* 2 *or* 3 *hold. If* $y(t)$ *is a solution of the Cauchy problem* (1), *which admits a bounded analytic extension to the sector* $S_\varphi := \{\lambda \in \mathbb{C}, |\arg\lambda| < \varphi\}$ *for some* $\varphi \in (0, \pi/2)$, *then there exist*

$$\lim_{t\to\infty} y(t) = z \quad \text{and} \quad z \in N(A).$$

The proof of Theorem 5 repeats the argument used to prove Theorem 2 (or Theorem 3, respectively). We need only to redefine $\mathfrak{N}'$ to be a set of all $w \in \mathfrak{B}$ such that there exists bounded solution $y(t)$, analytic in S_φ, of the Cauchy problem (1) with the initial value w. Then $\|w\|_{\mathfrak{N}} = \{\|x(t)\|, t \in S_\varphi\}$.

Corollary 5. *If the assumptions of one of Corollaries* 2 – 4 *hold, then the statement of Theorem* 5 *remains valid.*

Theorem 6. *In the statements of Theorems* 2 *and* 3, *we can replace* (C, α)*-limit by the Abel limit (for the definition of the* A*-limit, see* [1, 4]).

Proof is an immediate consequence of Theorems 2 and 3 and the lemma in [4, p. 92].

From Lemma 1, we can deduce the following corollary.

Corollary 6. *Let* $\mathfrak{B}$ *be a reflexive Banach space. If* $N(A) \cap R(A) = \{0\}$, *then*

$$\alpha t^{-\alpha} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \xrightarrow{w} z, \quad t \to \infty, \quad z \in N(A).$$

This fact is a generalization of Theorem 3 [5].

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