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# UNIQUENESS OF SOLUTION OF SOME NONLOCAL BOUNDARY-VALUE PROBLEMS FOR OPERATOR-DIFFERENTIAL EQUATIONS ON A FINITE SEGMENT*

# ЄДИНІСТЬ РОЗВ'ЯЗКУ ДЕЯКИХ НЕЛОКАЛЬНИХ КРАЙОВИХ ЗАДАЧ ДЛЯ ОПЕРАТОРНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ НА СКІНЧЕННОМУ ВІДРІЗКУ

For the equation $L_0x(t) + L_1x^{(1)}(t) + \ldots + L_nx^{(n)}(t) = 0$, where $L_k$, $k = 0, 1, \ldots, n$, are operators acting in a Banach space, we formulate criteria for a solution $x(t)$ to be zero, if it satisfies some nonlocal homogeneous boundary conditions.

Для рівняння $L_0x(t) + L_1x^{(1)}(t) + \ldots + L_nx^{(n)}(t) = 0$, де $L_k$, $k = 0, 1, \ldots, n$, — оператори, які діють у банаховому просторі, сформульовано умови рівності нулю розв'язку $x(t)$, що задовольняє деякі нелокальні однорідні крайові умови.

Assume that for Banach spaces $\mathcal{B}_k$, we have a chain of dense imbeddings (see, e. g., Ch. 1, § 1.1 [1])

$$\mathcal{B}_0 \to \mathcal{B}_1 \to \ldots \to \mathcal{B}_n.$$

In the sequel, all vector spaces are considered over the field of complex numbers **C**. Let $L_k$, $k = 0, 1, \ldots, n$, be bounded operators acting from the Banach space $\mathcal{B}_k$ to a Hilbert space $\mathcal{H}$ A vector-valued function $x(t)$ is called a weak solution of the equation

$$L_0x(t) + L_1x^{(1)}(t) + \ldots + L_nx^{(n)}(t) = 0 \qquad (1)$$

on the segment $[0; 1]$ if it is defined almost everywhere on $[0; 1]$, takes its values in $\mathcal{B}_0$, and satisfies the following conditions on $[0; 1]$:

(i) $x(t)$ is integrable in Pettis sense (see Sect. 3.7 [2]);

(ii) if a vector-valued function $x^{(l-1)}(t)$, $l = 1, 2, \ldots, n$, taking its values in $\mathcal{B}_{l-1}$ is given, then there exists almost everywhere a vector-valued function $x^{(l)}(t)$ whose values belong to $\mathcal{B}_l$, and, for each functional $f_l^* \in \mathcal{B}_l^*$, the function $f_l^*\left(x^{(l-1)}(t)\right)$ is absolutely continuous and

$$\frac{df_l^*\left(x^{(l-1)}(t)\right)}{dt} = f_l^*\left(x^{(l)}(t)\right) \quad \text{a. e. for} \quad t \in [0; 1];$$

(iii) $x(t)$, $x^{(1)}(t), \ldots, x^{(n)}(t)$ satisfies (1) almost everywhere.

Note that by the definition of solution of equation (1), the functions $x^{(l-1)}(t)$, $l = 1, 2, \ldots, n$, with values in $\mathcal{B}_{l-1}$ are defined almost everywhere in $[0; 1]$, and if they take values in $\mathcal{B}_l$, they are defined everywhere. This is why, in what follows, we regard values of $x^{(l-1)}(t)$ at the points $t = 0$ and $t = 1$ as vectors from space $\mathcal{B}_l$. The homogeneous Dirichlet problem for equation (1) is understood in the following sense:

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$$x^{(l-1)}(0) = 0, \quad l = 1, 2, \ldots, p, \quad x^{(l-1)}(1) = 0, \quad l = 1, 2, \ldots, q, \tag{2}$$

where $p \le n$ and $q \le n$.

In the next statements, we assume that for $p \le 0$ or $q \le 0$, conditions (2) are not imposed on $x(t)$ at the point 0 or 1, respectively.

Now we suppose that $\mathcal{B}_0 \to \mathcal{H}$ and consider the following nonlocal boundary conditions

$$\int_0^1 e^{-\lambda_\nu t} t^{s-1}(x(t), f)\, dt = 0, \quad f \in \mathcal{H}, \tag{3}$$

$$s = 1, 2, \ldots, d_\nu, \quad \nu = 1, 2, \ldots, r,$$

where $\lambda_\nu \in \mathbf{C}$ are fixed and $(\cdot, \cdot)$ is an inner product in $\mathcal{H}$.

Conditions (3) are well difined as long as $\mathcal{B}_0 \to \mathcal{H}$ and $x(t)$ takes values in $\mathcal{B}_0$.

With respect to the numbers $\lambda_\nu$ and $d_\nu$ from conditions (3), we introduce the polynomial

$$\rho(\lambda) := (\lambda - \lambda_1)^{d_1} \ldots (\lambda - \lambda_r)^{d_r},$$

the degree of which is $d = d_1 + \ldots + d_r$.

We assume later that for $d = 0$, the polynomial $\rho(\lambda) := 1$ and conditions (3) are not imposed on $x(t)$. Below, $\mathbf{R}$ is the set of real numbers; $[\alpha]$ is the integer part of $\alpha \in \mathbf{R}$ and

$$L(\lambda) = L_0 + \lambda L_1 + \ldots + \lambda^n L_n$$

is a bounded operator acting from $\mathcal{B}_0$ into $\mathcal{H}$ for each complex $\lambda$. This implies, in view of the imbedding $\mathcal{B}_0 \to \mathcal{H}$, that the inner product $(L(\lambda)x, x)$ is well difined for all vectors $x \in \mathcal{B}_0$.

**Theorem.** *Let*

$$\operatorname{Re} \rho(i\zeta)(L(i\zeta)x, x) \ge 0, \quad \zeta \in \mathbf{R}, \quad x \in \mathcal{B}_0, \tag{4}$$

*and*

$$\operatorname{Re} \rho(i\zeta_0)(L(i\zeta_0)x, x) > 0, \quad x \ne 0, \quad x \in \mathcal{B}_0, \tag{5}$$

*for some* $\zeta_0 \in \mathbf{R}$.

*Suppose that* $d \ge n$. *Then problem* (1), (3) *has only trivial solution.*

*Now suppose that* $d < n$ *and* $\mathcal{B}_{[(n-d+1)/n]} \to \mathcal{H}$. *Then problem* (1)–(3) *has only trivial solution in the following cases:*

(i) *if* $p = q = [(n-d+1)/2]$ *under conditions* (2);

(ii) *if* $n - d$ *is an odd number,* $(i)^{n+d-1}(L_n x, x) \le 0$ *for* $x \in \mathcal{B}_{(n-d+1)/2}$, *and* $p = (n-d-1)/2$, $q = (n-d+1)/2$ *under conditions* (2);

(iii) *if* $n - d$ *is an odd number,* $(i)^{n+d-1}(L_n x, x) \ge 0$ *for* $x \in \mathcal{B}_{(n-d+1)/2}$, *and* $p = (n-d+1)/2$, $q = (n-d-1)/2$ *under conditions* (2).

The proof of the Theorem uses the methods, which were developed in [3, 4].

***Remark.*** The Theorem shows that we can always reduce the number of homogeneous Dirichlet conditions (2) at the points $t = 0$ and $t = 1$ due to additions of conditions of kind (3). In fact, if condition (4) holds, it follows that

$$\operatorname{Re}\rho(i\zeta)(i\zeta-\lambda_{r+1})(i\zeta+\bar{\lambda}_{r+1})(-L(i\zeta)x,x) \ge 0, \quad \zeta\in\mathbf{R}$$

for al $x \in \mathcal{B}_0$. Here, $\lambda_{r+1}$ is an arbitrary complex number. Let $\lambda_{r+1} \ne i\zeta_0$, where $\zeta_0$ is the same as in condition (5). Then the condition of the Theorem holds with the polynomial $\rho(\lambda)(\lambda-\lambda_{r+1})(\lambda+\bar{\lambda}_{r+1})$, i. e., if the polynomial has degree $d+2$. Suppose that in the condition of the Theorem, $n \ge 2$ and $d \le n-2$. Then we can reduce, in each of the statements (i), (ii) and (iii) of the Theorem, the values $p$ and $q$ by one due to adding two conditions of kind (3) to the nonlocal conditions (3). For example, if $\lambda_{r+1} \ne \lambda_\nu$ and $-\bar{\lambda}_{r+1} \ne \lambda_\nu$ for $\nu = 1, 2, \ldots, r$, and $i\lambda_{r+1} \notin \mathbf{R}$, then the following conditions are added:

$$\int_0^1 e^{-\lambda_{r+1}t}(x(t),f)\,dt = \int_0^1 e^{\bar{\lambda}_{r+1}t}(x(t),f)\,dt = 0, \qquad f\in\mathcal{H}.$$

**Corollary 1.** *Let*

$$\mathcal{B}_{[(n+1)/2]} \to \mathcal{H}, \qquad \operatorname{Re}(L(i\zeta)x,x) \ge 0, \qquad \zeta\in\mathbf{R}, \quad x\in\mathcal{B}_0,$$

*and*

$$\operatorname{Re}(L(i\zeta_0)x,x) > 0 \quad \text{for all} \quad x\ne 0, \quad x\in\mathcal{B}_0, \quad \text{and some } \zeta_0\in\mathbf{R}.$$

*Then problem* (1), (2) *has only trivial solution in the following cases:*

(i) *if* $p = q = [(n+1)/2]$ *under conditions* (2);

(ii) *if* $n$ *is an odd number*, $(i)^{n-1}(L_n x, x) \le 0$ *for* $x\in\mathcal{B}_{(n+1)/2}$, *and* $p = (n-1)/2$ *and* $q = (n+1)/2$ *under conditions* (2);

(iii) *if* $n$ *is an odd number*, $(i)^{n-1}(L_n x, x) \ge 0$ *for* $x\in\mathcal{B}_{(n+1)/2}$, *and* $p = (n+1)/2$ *and* $q = (n-1)/2$ *under conditions* (2).

**Corollary 2.** *Let*

$$\mathcal{B}_{[n/2]} \to \mathcal{H}, \qquad \operatorname{Re} i\zeta(L(i\zeta)x,x) \ge 0, \qquad \zeta\in\mathbf{R}, \quad x\in\mathcal{B}_0,$$

*and*

$$\operatorname{Re} i\zeta_0(L(i\zeta_0)x,x) > 0 \quad \text{for all} \quad x\ne 0, \quad x\in\mathcal{B}_0, \quad \text{and some } \zeta_0\in\mathbf{R}.$$

*If the solution* $x(t)$ *of problem* (1), (2) *satisfies the requirement*

$$\int_0^1 (x(t),f)\,dt = 0, \qquad f\in\mathcal{H}, \tag{6}$$

*then* $x(t) = 0$, $0 \le t \le 1$, *in the following cases:*

(i) *if* $p = q = [n/2]$ *under conditions* (2);

(ii) *if* $n$ *is an even number*, $(i)^n(L_n x, x) \le 0$ *for* $x\in\mathcal{B}_{n/2}$, *and* $p = (n-2)/2$ *and* $q = n/2$ *under conditions* (2);

(iii) *if* $n$ *is an even number*, $(i)^n(L_n x, x) \ge 0$ *for* $x\in\mathcal{B}_{n/2}$, *and* $p = n/2$ *and* $q = (n-2)/2$ *under conditions* (2).

The requirement (6) coincides with the Neumann condition for solution $x(t)$ of equation (1). Therefore, conditions (3) are the Neumann generalized conditions.

The next example is a simple illustration of Corollary 2.

Let $\Omega$ be a bounded domain in $\mathbf{R}^s$ whose boundary $\partial\Omega$ is sufficiently smooth. Let also $c_1(\xi)$, $c_2(\xi)$, $c_3(\xi)$ be measurable essentially bounded functions. Denote by $A(\xi, \mathcal{D}_\xi)$ an elliptic differential operator on $\Omega$ of order $2m$, where $\mathcal{D}_\xi$ is the derivative with respect to the variable $\xi$. Let $B_j(\xi, \mathcal{D}_\xi)$, $j = 1, 2, \ldots, m$, denote a system of boundary differential operators. Suppose that

$$A(\xi, \mathcal{D}_\xi)\, x(\xi) = f(\xi), \quad \xi \in \Omega,$$

$$B_j(\xi, \mathcal{D}_\xi) x(\xi) = 0, \quad \xi \in \partial\Omega, \quad j = 1, 2, \ldots, m, \tag{7}$$

is a reqular elliptic problem (see Ch. 2, § 1.4 [5]). Consider the problem:

$$c_3(\xi)\frac{\partial^3 x(\xi, t)}{\partial t^3} + c_2(\xi)\frac{\partial^2 x(\xi, t)}{\partial t^2} + c_1(\xi)\frac{\partial x(\xi, t)}{\partial t} +$$

$$+ A(\xi, \mathcal{D}_\xi) x(\xi, t) = 0 \quad \text{a. e. for} \quad \xi \in \Omega, \quad t \in [0; 1], \tag{8}$$

$$B_j(\xi, \mathcal{D}_\xi) x(\xi, t) = 0 \quad \text{a. e. for} \quad \xi \in \partial\Omega, \quad j = 1, 2, \ldots, m, \quad t \in [0; 1], \tag{9}$$

$$x(\xi, 0) = x(\xi, 1) = 0 \quad \text{a. e. for} \quad \xi \in \Omega. \tag{10}$$

By a solution of the problem (8) – (10) we mean a function $x(\xi, t)$, $\xi \in \Omega$, $t \in [0; 1]$, whose derivatives taken in the sense of the theory of distributions

$$\mathcal{D}_\xi^k x(\xi, t), \quad |k| \le 2m, \qquad \frac{\partial^{l-1} x(\xi, t)}{\partial t^{l-1}}, \quad l = 1, 2, 3, 4,$$

belong to the tensor product $L_2(\Omega) \otimes L_1(0; 1)$ of two Lebesque spaces $L_2(\Omega)$ and $L_1(0; 1)$. If we use this and the imbedding theorem, we get that $B_j(\xi, \mathcal{D}_\xi) x(\xi, t)$ and $x(\xi, t)$ are defined almost everywhere on the boundary of domain $\Omega \times (0; 1)$. Therefore the equalities (9) and (10) hold true.

***Corollary 3.*** *Let*

$$\operatorname{Re} c_1(\xi) \ge 0, \quad \operatorname{Im} c_2(\xi) = 0, \quad \operatorname{Re} c_3(\xi) \le 0, \quad \operatorname{Re}(c_1(\xi) - c_3(\xi)) > 0,$$

*almost everywhere for* $\xi \in \Omega$, *and the problem* (7) *is formal self-adjoint. Suppose that the solution* $x(\xi, t)$ *of problem* (8) – (10) *satisfies the condition:* $\int_0^1 x(\xi, t)\, dt = 0$ *almost everywhere for* $\xi \in \Omega$. *Then* $x(\xi, t) = 0$ *almost everywhere for* $\xi \in \Omega$ *and* $t \in [0; 1]$.


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