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GENERATORS AND RELATIONS FOR WREATH PRODUCTS
ТВІРНІ ТА СПІВВІДНОШЕННЯ ДЛЯ ВІНЦЕВИХ ДОБУТКІВ

Generators and defining relations for wreath products of groups are given. Under a certain condition (conormality of generators), they are minimal.

Наведено твірні та визначальні співвідношення для вінцевих добутків. За деякої умови (конормальність твірних) вони є мінімальними.

Let G, H be two groups. Denote by H^G the group of all maps $f\colon G \to H$ with *finite support*, i.e., such that $f(x) = 1$ for all but a finite set of elements of G. Recall that their (*restricted regular*) *wreath product* $W = H \wr G$ is defined as the semidirect product $H^G \rtimes G$ with the natural action of G on $H^G : f^g(a) \to f(ag)$ [1, p. 175]. We are going to find a set of generators and relations for $H \wr G$ knowing those for G and H. Then we shall extend this result to the *multiple wreath products* $\wr_{k=1}^{n} G_k = (\ldots((G_1 \wr G_2) \wr G_3)\ldots) \wr G_n$.

If $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ are generators for G and $\mathbf{R} = \{R_1, R_2, \ldots, R_m\}$ are defining relations for this set of generators, we write $G := \langle x_1, x_2, \ldots, x_n \mid R_1, R_2, \ldots, R_m \rangle$ or $G := \langle \mathbf{x} \mid \mathbf{R} \rangle$. A presentation is called *minimal* if neither of the generators $x_1, x_2, \ldots, x_n$ nor of the relations $R_1, R_2, \ldots, R_m$ can be excluded. We call the set of generators $\mathbf{x}$ *conormal* if neither element $x \in \mathbf{x}$ belongs to the normal subgroup N_x generated by all $y \in \mathbf{x} \setminus \{x\}$. For instance, any minimal set of generators of a finite p-group G is conormal since their images are linear independent in the factorgroup $G/G^p[G, G]$ [1] (Theorem 5.48).

Theorem 1. *Let* $G := \langle \mathbf{x} \mid \mathbf{R}(\mathbf{x}) \rangle$, $H := \langle \mathbf{y} \mid \mathbf{S}(\mathbf{y}) \rangle$ *be presentations of* G *and* H. *Choose a subset* $T \subseteq G$ *such that* $T \cap T^{-1} = \varnothing$ *and* $T \cup T^{-1} = G \setminus \{1\}$, *where* $T^{-1} = \{t^{-1} \mid t \in T\}$. *Then the wreath product* $W = H \wr G$ *has a presentation of the form*

$$W := \langle \mathbf{x}, \mathbf{y} \mid \mathbf{R}(\mathbf{x}), \mathbf{S}(\mathbf{y}), [y, t^{-1} z t] = 1 \quad \text{for all } y, z \in \mathbf{y},\ t \in T \rangle. \tag{1}$$

If the given presentations of G *and* H *are minimal and the set of generators* $\mathbf{y}$ *is conormal, the presentation* (1) *is minimal as well.*

Theorem 2. *Let* $G_i := \langle \mathbf{x}_i \mid \mathbf{R}_i(\mathbf{x}_i) \rangle$ *be presentations of the groups* G_i, $1 \le i \le m$. *For* $1 < i \le m$ *choose a subset* $T_i \subseteq G_i$ *such that* $T_i \cap T_i^{-1} = \varnothing$ *and* $T_i \cup T_i^{-1} = G_i \setminus \{1\}$. *Then the wreath product* $W = \wr_{i=1}^{n} G_i$ *has a presentation of the form*

$$W := \langle \mathbf{x}_i, 1 \le i \le m \mid \mathbf{R}_i(\mathbf{x}_i), 1 \le i \le m, [x, t^{-1} y t] = 1 \quad \text{for all } x, y \in \textstyle\bigcup_{i<j} \mathbf{x}_i,\ t \in T_j \rangle. \tag{2}$$

If all given presentations of G_i *are minimal and the sets of generators* $\mathbf{x}_i$, $1 \le i < n$, *are conormal, the presentation* (2) *is minimal as well.*

In what follows, we keep the notations of Theorem 1. Note that $H^G = \oplus_{a\in G} H(a)$, where $H(a)$ is a copy of the group H; the elements of $H(a)$ will be denoted by $h(a)$, where h runs through H. Then $h(a)^g = h(ag)$ and $H^G = \langle \mathbf{y}(a) \,|\, \mathbf{S}(\mathbf{y}(a)), [y(a), z(b)] = 1\rangle$, where $a, b \in G$, $a \neq b$.

The following lemma is quite evident.

Lemma 1. *Suppose a group G acting on a group N. Let $G = \langle \mathbf{x} \,|\, \mathbf{R}(\mathbf{x})\rangle$, $N = \langle \mathbf{y} \,|\, \mathbf{S}(\mathbf{y})\rangle$ be presentations of G and N, and $y^x = w_{xy}(\mathbf{y})$ for each $x \in \mathbf{x}$, $y \in \mathbf{y}$. Then their semidirect product $N \rtimes G$ has a presentation*

$$N \rtimes G := \langle \mathbf{x}, \mathbf{y} \,|\, \mathbf{R}(\mathbf{x}), \mathbf{S}(\mathbf{y}), x^{-1}yx = w_{xy}(\mathbf{y}) \quad \text{for all } x \in \mathbf{x},\ y \in \mathbf{y}\rangle.$$

Note that this presentation may not be minimal even if both presentations for G and N were so, since some elements of $\mathbf{y}$ may become superfluous.

***Corollary* 1.** *The wreath product $W = H \wr G$ has indeed a presentation* (1).

Proof. Lemma 1 gives a presentation

$$W := \langle \mathbf{x}, \mathbf{y}(a) \,|\, \mathbf{R}(\mathbf{x}), \mathbf{S}(\mathbf{y}(a)), [y(a), z(b)] = 1$$

$$x^{-1}y(a)x = y(ax) \quad \text{for } x \in \mathbf{x},\ y, z \in \mathbf{y},\ a, b \in G,\ a \neq b\rangle.$$

Using the last relations, we can exclude all generators $y(a)$ for $a \neq 1$; we only have to replace $y(a)$ and $z(b)$ by $a^{-1}y(1)a$ and $b^{-1}z(1)b$. So we shall write h instead of $h(1)$ for $h \in H$; especially, the relations for $y(a)$ and $z(b)$ are rewritten as $[a^{-1}ya, b^{-1}zb] = 1$. The latter is equivalent to $[y, t^{-1}zt] = 1$, where $t = ba^{-1} \neq 1$. Moreover, the relations $[y, t^{-1}zt] = 1$ and $[z, tyt^{-1}] = 1$ are also equivalent; therefore we only need such relations for $t \in T$.

The corollary is proved.

Lemma 2. *Suppose that $\mathbf{y}$ is a conormal set of generators of the group H, u, $v \in \mathbf{y}$, and consider the group $H_{u,v} = (H * H')/N_{u,v}$, where $*$ denotes the free product of groups, H' is a copy of the group H whose elements are denoted by h' ($h \in H$), and $N_{u,v}$ is the normal subgroup of $H * H'$ generated by the commutators $[y, z']$ with $y, z \in \mathbf{y}$, $(y, z) \neq (u, v)$. Then $[u, v'] \neq 1$ in $H_{u,v}$.*

Proof. Let $C = H/N_u$, $C' = H'/N_{v'}$, $P = C * C'$, $\bar{u} = uN_u$, $\bar{v} = v'N_{v'}$. Consider the homomorphism φ of $H * H'$ to P such that

$$\varphi(y) = \begin{cases} 1 & \text{if } y \in \mathbf{y}_u, \\ \bar{u} & \text{if } y = u, \end{cases}$$

$$\varphi(z') = \begin{cases} 1 & \text{if } z \in \mathbf{y}_v, \\ \bar{v}' & \text{if } z = v. \end{cases}$$

Obviously, φ is well defined and $\varphi([y, z']) = 1$ if $(y, z) \neq (u, v)$, so it induces a homomorphism $H_{u,v} \to P$. Since $\varphi([u, v']) = [\bar{u}, \bar{v}'] \neq 1$, it accomplishes the proof.

Now fix elements $c \in T$, $u, v \in \mathbf{y}$, and let $K_{c,u,v}$ be the group with a presentation

$$K_{c,u,v} := \langle \mathbf{y}(a),\ a \in G \,|\, \mathbf{S}(\mathbf{y}(a)), [y(a), z(ta)] = 1$$

$$\text{for all } y, z \in \mathbf{y},\ a \in G,\ t \in T,\ (t, y, z) \neq (c, u, v)\rangle.$$

***Corollary* 2.** *Let the set of generators* $\mathbf{y}$ *be conormal. Then* $[u(1), v(c)] \neq 1$ *in the group* $K_{c,u,v}$.

Proof. There is a homomorphism $\psi : K_{c,u,v} \to H_{u,v}$, where $H_{u,v}$ is the group from Lemma 2, mapping $u(1) \mapsto u$, $v(c) \mapsto v'$, $y(a) \to 1$ in all other cases. Then $\psi([u(1), v(c)]) = [u, v'] \neq 1$, so $[u(1), v(c)] \neq 1$ as well.

***Corollary* 3.** *If the given presentations of* G *and* H *are minimal and the set of generators* $\mathbf{y}$ *is conormal, the presentation* (1) *is minimal.*

Proof. Obviously, we can omit from (1) neither of generators $\mathbf{x}$, $\mathbf{y}$ nor of the relations $\mathbf{R}(\mathbf{x})$, $\mathbf{S}(\mathbf{y})$. So we have to prove that neither relation $[u, c^{-1}vc] = 1$ $(u, v \in \mathbf{y}, c \in T)$ can be omitted as well. Consider the group $K = K_{c,u,v}$ of Corollary 2. The group G acts on K by the rule: $h(a)^g = h(ag)$. Let $Q = K \rtimes G$. Then, just as in the proof of Corollary 1, this group has a presentation

$$Q := \langle \mathbf{x}, \mathbf{y} \,|\, \mathbf{R}(\mathbf{x}), \mathbf{S}(\mathbf{y}), [y, t^{-1}zt] = 1 \quad \text{for all} \quad y, z \in \mathbf{y},\ t \in T,\ (t, y, z) \neq (c, u, v) \rangle,$$

where $y = y(1)$ for all $y \in \mathbf{y}$, but $[u, c^{-1}vc] = [u(1), v(c)] \neq 1$.

The corollary is proved.

Now for an inductive proof of Theorem 2 we only need the following simple result.

Lemma 3. *If the sets of generators* $\mathbf{x}$ *of* G *and* $\mathbf{y}$ *of* H *are conormal, so is the set of generators* $\mathbf{x} \cup \mathbf{y}$ *of* $H \wr G$.

Proof. Since $G \simeq (H \wr G)/\hat{H}$, where $\hat{H}$ is the normal subgroup generated by all $y \in \mathbf{y}$, it is clear that neither $x \in \mathbf{x}$ belongs to the normal subgroup generated by $(\mathbf{x} \setminus \{x\}) \cup \mathbf{y}$. On the other hand, there is an epimorphism $H \wr G \to C \wr G$, where $C = H/N_y$ for some $y \in \mathbf{y}$; in particular, $C \neq \{1\}$ and is generated by the image $\bar{y}$ of y. Since C is commutative, the map $C \wr G \to C$, $(f(x), g) \mapsto \prod_{x \in G} f(x)$ is also an epimorphism mapping $\bar{y}$ to itself. The resulting homomorphism $H \wr G \to C$ maps all $x \in \mathbf{x}$ as well as all $z \in \mathbf{y} \setminus \{y\}$ to 1 and y to $\bar{y} \neq 1$, which accomplishes the proof.

***Example* 1.** The wreath product $C_n \wr C_m$, where C_n denotes the cyclic group of order n, has a minimal presentation

$$C_n \wr C_m := \langle x, y \,|\, x^m = 1,\ y^n = 1,\ [y, x^{-k} y x^k] = 1 \quad \text{for} \quad 1 \leq k \leq m/2 \rangle.$$

(Possibly, $m = \infty$ or $n = \infty$, then the relation $x^m = 1$ or, respectively, $y^n = 1$ should be omitted.)

1. *Rotman J. J.* An introduction to the theory of groups. – New York: Springer, 1995. – XV + 513 p.
2. *Kaloujnine L. A.* Sur les p-group de Sylow du groupe symétrique du degre p^m // C. r. Acad. sci. Paris. – 1945. – **221**. – P. 222 – 224.

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