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CONTINUITY WITH RESPECT TO THE INITIAL DATA AND ABSOLUTE-CONTINUITY APPROACH TO THE FIRST-ORDER REGULARITY OF NONLINEAR DIFFUSIONS ON NONCOMPACT MANIFOLDS

НЕПЕРЕРВНІСТЬ ЗА ПОЧАТКОВИМИ УМОВАМИ ТА ПІДХІД ТЕОРІЇ АБСОЛЮТНО НЕПЕРЕРВНИХ ФУНКЦІЙ ДО РЕГУЛЯРНОСТІ ПЕРШОГО ПОРЯДКУ ДЛЯ НЕЛІНІЙНИХ ДИФУЗІЙ НА НЕКОМПАКТНИХ РІМАНОВИХ БАГАТОВИДАХ

We study the dependence with respect to the initial data for solutions of diffusion equations with globally non-Lipschitz coefficients on noncompact manifolds. Though the metric distance may be not everywhere twice differentiable, we show that under some monotonicity conditions on coefficients and curvature of manifold there are estimates exponential in time on the continuity of diffusion process with respect to the initial data.

These estimates are combined with methods of the theory of absolutely continuous functions to achieve the first-order regularity of solutions with respect to the initial data. The suggested approach neither appeals to the local stopping time arguments, nor applies the exponential mappings on tangent space, nor uses embeddings of manifold to linear spaces of higher dimensions.

Досліджено залежність за початковими умовами для розв'язків дифузійних рівнянь з глобально неліпшицевими коефіцієнтами на некомпактних багатовидах. Хоча функція метричної відстані може бути не скрізь двічі диференційовною, показано, що за певних умов монотонності на коефіцієнти та кривину багатовиду існують експоненціальні за часом оцінки на неперервність дифузійного процесу за початковими умовами.

У поєднанні з методами теорії абсолютно неперервних функцій ці оцінки приводять до першого порядку регулярності розв'язків за початковими умовами. Запропонований підхід не використовує техніку моментів часу виходу процесу з локальних координатних околів, а також експоненціальних відображень з дотичного простору або вкладення багатовиду до лінійного простору більшої розмірності.

1. Introduction. In this article we study the continuous dependence and the first-order regularity with respect to the initial data for Ito – Stratonovich diffusion

$$\delta y_t^x = A_0(y_t^x)dt + \sum_{\alpha=1}^{d} A_\alpha(y_t^x)\delta W_t^\alpha, \quad y_0^x = x, \tag{1.1}$$

on noncompact oriented smooth complete connected Riemannian manifold M without boundary. Here A_0, A_α, $\alpha = 1, \ldots, d$, are globally defined C^∞-smooth vector fields on M and δW^α denote Stratonovich differentials of one dimensional independent Wiener processes W_t^α, $\alpha = 1, \ldots, d$.

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Under the solution of (1.1) it is understood a continuous adapted process y_t^x such that for any C^∞-function f with a compact support on manifold M the following stochastic integral equation is satisfied

$$f(y_t^x) = f(x) + \int_0^t (A_0 f)(y_s^x)ds + \sum_{\alpha=1}^d \int_0^t (A_\alpha f)(y_s^x)\delta W_s^\alpha. \tag{1.2}$$

Traditionally the continuity of diffusion process y_t^x with respect to the initial data, i.e., estimate

$$\exists K_p\colon\ \mathbf{E}\rho^p(y_t^x, y_t^z) \le e^{K_p t}\rho^p(x,z), \tag{1.3}$$

is obtained using the geodesic deviations formulas and Jacobi fields approach. The naturally arising conditions are related with the global Lipschitz assumptions on coefficients of diffusion equation and semiboundedness of curvature of manifold, e.g. [1 – 5].

It is known that under the same global Lipschitz assumptions and semiboundedness of curvature the process y_t^x is first-order regular with respect to the initial data x and there is derivative $\dfrac{\partial y_t^x}{\partial x}$, e.g. [1] (Ch. 4, § 3), [2] (Ch. VIII). The restriction on curvature is related with the use of uniform exponential charts of manifold M with further estimation of local difference expressions for derivatives

$$\frac{y_t^{x+\varepsilon h} - y_t^x}{\varepsilon} - \frac{\partial y_t^x}{\partial x}[h]. \tag{1.4}$$

Later in [6] (Ch. 4, § 8), by application of local stopping time techniques, it was demonstrated that the mapping $M \ni x \to y_t^x \in M$ represents a diffeomorphism till the first explosion time. However, it is still not clear, what global assumptions on coefficients and curvature, besides global Lipschitzness and semiboundedness of curvature, may lead to non-explosion and first-order regularity of nonlinear diffusion processes on noncompact manifolds.

In [7] it was found a way to avoid the application of geodesic deviation's techniques. In [8] these results were used to prove the non-explosion of nonlinear diffusion y_t^x, i.e., the existence and uniqueness of solutions of (1.1) for all $t \ge 0$. The proposed conditions on the coefficients of diffusion equation and curvature represent a manifold analogue of coercitivity and dissipativity conditions, known previously for nonlinear monotone equations on linear spaces [9, 10].

In this article we prove that under the same conditions process y_t^x continuously depends on the initial data x. Moreover, using the methods of absolute continuous functions theory, we demonstrate that the existence of first-order derivative $\dfrac{\partial y_t^x}{\partial x}$ with respect to the initial data is a direct consequence of the continuity estimates (1.3).

In Section 2 we formulate the main result of the article. Sections 3 and 4 are devoted to the proof of continuous dependence of diffusion process y_t^x with respect to the initial data x. In comparison to the Euclidean space with C^∞-smooth square of metric distance $\rho^2(x,y) = \|x-y\|^2$, for the general manifold the square of metric distance $\rho^2(x,y)$, defined as a shortest geodesic distance

$$\rho^2(x,z) = \inf\left\{ \int_0^1 |\dot\gamma(\ell)|^2 d\ell,\ \ \gamma(0)=x,\ \ \gamma(1)=z \right\} \quad \text{for} \quad \dot\gamma(\ell) = \frac{\partial}{\partial\ell}\gamma(\ell), \tag{1.5}$$

may be not twice differentiable everywhere, i.e., the following second-order operator:

$$\mathcal{L}f(x,z) = \left\{ A_0^I(x) + A_0^{II}(z) + \frac{1}{2}\sum_{\alpha=1}^{d}\Big(A_\alpha^I(x) + A_\alpha^{II}(z)\Big)^2 \right\} f(x,z)$$

with $A^I(x)$, $A^{II}(z)$ acting correspondingly on the first x and second z variables of function $f(x,z)$, may be undefined.

Therefore, the direct application of Ito formula

$$\mathbf{E}\rho^p(y_t^x, y_t^z) = \rho^p(x,z) + \int_0^t \mathbf{E}(\mathcal{L}\rho^p)(y_s^x, y_s^z)ds \tag{1.6}$$

in order to obtain (1.3) from upper bounds

$$\mathcal{L}\rho^p(x,z) \le K_p \rho^p(x,z) \tag{1.7}$$

becomes impossible.

In Section 3 we develop results of [7, 8] and replace the strong estimates (1.7) by the weak estimates on operators of structure (1.6), acting on the metric function on the product of manifolds $M \times M$. The main difference from [8] is that we have to work with two point functions $\rho(x,z)$ instead of estimation of $\rho(o,x)$ for some fixed $o \in M$.

In Section 4, following the arguments of [11], we apply weak estimates on operators $\mathcal{L}$ to show that for coercitive and dissipative diffusion there is a constant K such that process

$$\rho^2(y_t^x, y_t^z) - K\int_0^t \rho^2(y_s^x, y_s^z)ds$$

represents a supermartingale. This, in fact, replaces the Ito formula approach (1.6) and leads to (1.3).

In Section 5 we demonstrate that the solution $y_t^{(1)}(x)$ of the first-order variational equation represents the first-order derivative $y_t^{(1)}(x) = \dfrac{\partial y_t^x}{\partial x}$ with respect to the initial data.

First we construct special coordinate systems $x_i = \rho(o_i, x)$ in small local vicinities of manifold. The use of these particular coordinates and special transfer of relation (1.2) from manifold M to $I\!R^{\dim M}$ permit to obtain the first-order regularity from continuity estimates (1.3).

Namely, estimate (1.3) implies that for Lipschitz continuous path $[a,b] \ni u \to$ $\to h(u) \in M$ on manifold the mapping

$$[a,b] \ni u \to y_t^{h(u)} \tag{1.8}$$

is Lipschitz continuous, i.e., for a.e. $u \in [a,b]$ there is derivative $\theta_t(u) = \dfrac{dy_t^{h(u)}}{du}$ with $|\theta_t(u)| \in L^\infty\big([a,b]\times[0,T], L^p(\Omega)\big)$ for all $T > 0$, $p \ge 1$. By definition of solution (1.2) we have for $f \in C_0^\infty(M)$

$$\int_a^b \Big\langle \nabla f(y_t^{h(u)}), \theta_t(u)\Big\rangle du = f(y_t^{h(b)}) - f(y_t^{h(a)}) =$$

$$= f(h(b)) - f(h(a)) + \int_0^t \Big[(A_\alpha f)(y_s^{h(b)}) - (A_\alpha f)(y_s^{h(a)})\Big]\delta W_s^\alpha +$$

$$+ \int_0^t \Big[(A_0 f)(y_s^{h(b)}) - (A_0 f)(y_s^{h(a)})\Big] ds =$$

$$= \int_a^b \Big\langle \nabla f(h(u)), h'(u)\Big\rangle du + \int_0^t \left[\int_a^b \Big\langle \nabla (A_\alpha f)(y_s^{h(u)}), \theta_s(u)\Big\rangle du\right] \delta W_s^\alpha +$$

$$+ \int_0^t \left[\int_a^b \Big\langle \nabla (A_0 f)(y_s^{h(u)}), \theta_s(u)\Big\rangle du\right] ds. \tag{1.9}$$

Removing the integral $\int_a^b \ldots ds$ we obtain equation on the first-order variation for process $\theta_t(u)$. Therefore the derivative with respect to the initial data $\theta_t(u)$ must represent solution to the first-order variational equation $\theta_t(u) = y_t^{(1)}(h(u))$. In Section 5 formal reasoning (1.8), (1.9) is made rigorous.

Finally remark that the use of arbitrary paths $h \in \mathrm{Lip}\big([a,b], M\big)$ permits to avoid the separate conditions on the curvature of manifold, related with the existence of uniform exponential charts, like in [1, 2]. Moreover, the application of absolute continuous functions theory has also given a possibility to avoid the use of pure stochastic techniques of local stopping times, necessary for the estimation of difference expressions (1.4) in local coordinate systems, like in [4, 6]. Actually, we demonstrate that the first-order regularity is a direct consequence of continuity estimates (1.3).

2. Main result. Let us suppose that the coefficients of equation (1.1) and curvature tensor of manifold fulfill the following assumptions:

coercitivity: $\exists\, o \in M$ such that $\forall C \in \mathbb{R}_+ \;\exists\, K_C \in \mathbb{R}^1$ such that $\forall x \in M$

$$\Big\langle \widetilde{A_0}(x), \nabla^x \rho^2(o,x)\Big\rangle + C\sum_{\alpha=1}^d \|A_\alpha(x)\|^2 \le K_C(1+\rho^2(o,x)); \tag{2.1}$$

dissipativity: $\forall C, C' \in \mathbb{R}_+ \;\exists\, K_C \in \mathbb{R}^1$ such that $\forall x \in M$, $\forall h \in T_x M$

$$\Big\langle \nabla\widetilde{A_0}(x)[h], h\Big\rangle + C\sum_{\alpha=1}^d \big\|\nabla A_\alpha(x)[h]\big\|^2 -$$

$$- C'\sum_{\alpha=1}^d \Big\langle R_x(A_\alpha(x), h)A_\alpha(x), h\Big\rangle \le K_C\|h\|^2, \tag{2.2}$$

where $\widetilde{A_0} = A_0 + \frac{1}{2}\sum_{\alpha=1}^d \nabla_{A_\alpha} A_\alpha$ and notation $\nabla H[h]$ means *covariant derivative in direction* h

$$\big(\nabla H(x)[h]\big)^i = \nabla_j H^i(x)\cdot h^j. \tag{2.3}$$

and $\big[R(A,h)A\big]^m = R_{p\ \ell q}^{\ m} A^p A^\ell h^q$ for (1,3) — curvature tensor with components

$$R_{p\ \ell q}^{\ m} = \frac{\partial \Gamma_{p\ \ell}^{\ m}}{\partial x^q} - \frac{\partial \Gamma_{p\ q}^{\ m}}{\partial x^\ell} + \Gamma_{p\ \ell}^{\ j}\Gamma_{j\ q}^{\ m} - \Gamma_{p\ q}^{\ j}\Gamma_{j\ \ell}^{\ m}; \tag{2.4}$$

nonlinear behaviour of coefficients and curvature: for any n there are constants $\mathbf{k}_0$, $\mathbf{k}_\alpha$, $\mathbf{k}_R$ such that for all $j = 1,\ldots,n$ and $\forall\, x \in M$:

$$\begin{gathered} \|(\nabla)^j \widetilde{A_0}(x)\| \leq (1+\rho(x,z))^{\mathbf{k}_0}, \\ \|(\nabla)^j A_\alpha(x)\| \leq (1+\rho(x,z))^{\mathbf{k}_\alpha}, \\ \|(\nabla)^j R(x)\| \leq (1+\rho(x,z))^{\mathbf{k}_R}. \end{gathered} \tag{2.5}$$

Denote by $\mathrm{Lip}\big([a,b],M\big)$ the space of Lipschitz paths on $[a,b]$ with values in M. It is formed from continuous paths $h \in C\big([a,b],M\big)$ such that $\exists K_h\ \forall\, c,d \in [a,b]$ there is estimate on metric distance $\rho(h(c),h(d)) \leq K_h|c-d|$. In particular, by theory of absolute continuous functions this means that $\|h'\| \in L^\infty\big([a,b],TM\big)$ and Lipschitzness constant $K_h = \sup\limits_{z\in[a,b]} \|h'(z)\|_{T_{h(z)}M}$.

Theorem 2.1. *Under conditions* (2.1), (2.2), *and* (2.5) *the solution y_t^x of diffusion equation* (1.1) *is differentiable with respect to the initial data.*

Its derivative $y_t^{(1)}(x) = \dfrac{\partial y_t^x}{\partial x}$ represents a unique solution to the first-order variational equation, written in local coordinates

$$\begin{gathered} \delta[y_t^{(1)}(x)]_k^m = -\Gamma_{p\ q}^{\ m}(y_t^x)\left[(y_t^{(1)}(x)\right]_k^p \delta(y_t^x)^q + \nabla_p A_\alpha^m(\xi_t^x)\big[y_t^{(1)}(x)\big]_k^p \delta W_t^\alpha + \\ + \nabla_p A_0^m(\xi_t^x)\big[y_t^{(1)}(x)\big]_k^p dt \end{gathered} \tag{2.6}$$

with initial data $y_0^{(1)}(x) = I$ given by identity matrix.

Moreover, for any path $h \in \mathrm{Lip}([a,b],M)$ and $f \in C_0^\infty(M)$ a.e. integral relation is true:

$$f(y_t^{h(b)}) - f(y_t^{h(a)}) = \int\limits_a^b \Big\langle \nabla f(y_t^{h(z)}), y_t^{(1)}(h(z))\big[h'(z)\big]\Big\rangle_{T_{y_t^{h(z)}}} dz. \tag{2.7}$$

Under solution of (2.6) *it is understood a continuous adapted integrable process*

$$\mathbb{R}_+ \times M \ni (t,x) \ \longrightarrow$$

$$\longrightarrow\ y_t^{(1)}(x) \in L^\infty\big([0,T], L^p(\Omega, T_{y_t^x}M \otimes T_x^*M)\big), \quad \textit{for all} \quad T>0, \quad p>1,$$

such that for any $f \in C_0^\infty(M)$, $h \in T_xM$

$$\begin{gathered} \Big\langle \nabla f(y_t^x), y_t^{(1)}(x)[h]\Big\rangle_{T_{y_t^x}M} = \Big\langle \nabla f(x), y_0^{(1)}(x)[h]\Big\rangle_{T_xM} + \\ + \int\limits_0^t \Big\langle \nabla(A_\alpha f)(y_s^x), y_t^{(1)}(x)[h]\Big\rangle \delta W_s^\alpha + \int\limits_0^t \Big\langle \nabla(A_0 f)(y_s^x), y_t^{(1)}(x)[h]\Big\rangle ds. \end{gathered} \tag{2.8}$$

The proof is conducted in further sections.

3. Weak estimates on diffusion generators. Consider open set $U \subseteq M$ with compact closure $\overline{U}$ and function ζ^U such that $\sqrt{\zeta^U} \in C_0^\infty(M, [0,1])$ and $\zeta^U(z) = 1$ for $z \in \overline{U}$, $0 \leq \zeta^U < 1$ otherwise. Consider differential operator on $M \times M$

$$\mathcal{L}^U f(x,z) = \zeta^U(x)\zeta^U(z)\mathcal{L}f(x,z).$$

Being a second-order differential operator with localized coefficients, $\mathcal{L}^U$ corresponds to the localized Stratonovich diffusion $y_t^U(x,z)$ on $M \times M$

$$\delta y_t^U(x,z) = \sum_{\alpha=1}^d \sqrt{\zeta^U(y_t^{I,U})}\sqrt{\zeta^U(y_t^{II,U})}\left\{A_\alpha^I(y_t^{I,U}) + A_\alpha^{II}(y_t^{II,U})\right\}\delta W^\alpha +$$

$$+ \zeta^U(y_t^{II,U})\Bigg\{\zeta^U(y_t^{I,U}A_0^I(y_t^{I,U}) -$$

$$- \frac{1}{2}\sqrt{\zeta^U(y_t^{I,U})}\sum_{\alpha=1}^d (A_\alpha\sqrt{\zeta^U})(y_t^{I,U})A_\alpha^I(y_t^{I,U})\Bigg\}dt +$$

$$+\zeta^U(y_t^{I,U})\Bigg\{\zeta^U(y_t^{II,U}A_0^{II}(y_t^{II,U}) -$$

$$- \frac{1}{2}\sqrt{\zeta^U(y_t^{II,U})}\sum_{\alpha=1}^d (A_\alpha\sqrt{\zeta^U})(y_t^{II,U})A_\alpha^{II}(y_t^{II,U})\Bigg\}dt, \quad (3.1)$$

with initial data $y_0^U(x,z) = (x,z)$, where $y_t^{I,U}$ and $y_t^{II,U}$ are first and second components of process $y_t^U = (y_t^{I,U},\ y_t^{II,U})$ on the product $M \times M$. Remark that, due to property $\zeta^U\big|_U = 1$, for initial data $x, z \in U$ process $y_t^U(x,z)$ coincides with process (y_t^x, y_t^z) till the first exit time $t \leq \tau(\omega) = \inf\left\{t\colon\ y_t^x(\omega) \notin U \text{ or } y_t^z(\omega) \notin U\right\}$.

Equation (3.1) has globally Lipschitz coefficients with all bounded derivatives, therefore it has unique solution that C^∞-regularly depends on the initial data x [4–6, 11]. Its diffusion semigroup $(P_t^U f)(x) = \mathbf{E}f(y_t^U(x,z))$ preserves the space $C_{0,+}^\infty(M \times M)$ of non-negative C^∞-functions with compact support.

Main result of this section lies in the weak uniform with respect to U estimates on generators $\mathcal{L}^U$.

Theorem 3.1. *Under conditions* (2.1), (2.2), *and* (2.5) *there is* K *such that* $\forall\, \zeta^U \in C_0^\infty(M,[0,1])$ *with* $\zeta^U\big|_{\overline{U}} = 1$ *and* $\forall \varphi \in C_{0,+}^\infty(M \times M)$

$$\int\limits_{M\times M} ([\mathcal{L}^U]^*\varphi)\rho^2(x,z)d\sigma(x)d\sigma(z) \leq K \int\limits_{M\times M} \varphi(x,z)\rho^2(x,z)d\sigma(x)d\sigma(z). \quad (3.2)$$

Proof. As $[\mathcal{L}^U]^* = [\zeta^U(x)\zeta^U(z)\mathcal{L}]^* = \mathcal{L}^*\zeta^U(x)\zeta^U(z)$, estimate (3.2) will follow from the weak estimate on operator $\mathcal{L}$: $\exists K\ \forall \psi \in C_{0,+}^\infty(M \times M)$

$$\int\limits_{M\times M} (\mathcal{L}^*\psi(x,z))\rho^2(x,z)d\sigma(x)d\sigma(z) \leq K \int\limits_{M\times M} \psi(x,z)\rho^2(x,z)d\sigma(x)d\sigma(z) \quad (3.3)$$

if one substitutes first $\psi(x,z) = \zeta^U(x)\zeta^U(z)\varphi(x,z)$ and then applies $0 \leq \zeta^U \leq 1$.

Similar to [8] (ff. (16) – (18)) the following representation for the left-hand side of (3.3) is fulfilled

$$\int\limits_{M\times M} (\mathcal{L}^*\psi(x,z))\rho^2(x,z)d\sigma(x)d\sigma(z) =$$

$$= \lim_{\varepsilon\to 0+} \int\limits_{M\times M} \psi(x,z)\Bigg\{\frac{\rho^2(\eta_0^\varepsilon(x),\eta_0^\varepsilon(z))-\rho^2(x,z)}{\varepsilon}+$$

$$+\frac{1}{2}\sum_{\alpha=1}^{d}\frac{\rho^2(\eta_\alpha^\varepsilon(x),\eta_\alpha^\varepsilon(z))+\rho^2(\eta_\alpha^{-\varepsilon}(x),\eta_\alpha^{-\varepsilon}(z))-2\rho^2(x,z)}{\varepsilon^2}\Bigg\}d\sigma(x)d\sigma(z). \tag{3.4}$$

Here η_0^ε, η_α^ε denote the shifts along vector fields A_0, A_α and $\eta_\cdot^0(x) = x$. Operator $\mathcal{L}^*$ has representation $\mathcal{L}^* = \dfrac{1}{2}\sum_{\alpha=1}^{d}(A_\alpha^*)^2 + A_0^*$ in terms of adjoint fields $X^*f = -(\operatorname{div} X)f - Xf$ to vector field X.

Now let us estimate fractions in the right-hand side of (3.4). In the vicinity of geodesic $\gamma(\ell)$, $\ell\in[0,1]$ from $\gamma(0) = x$ to $\gamma(1) = z$ that minimizes (1.5) consider smooth vector field H. Introduce a family of paths

$$[0,1]\times(-\delta,\delta)\ni(\ell,s)\to\gamma(\ell,s)\in M$$

such that at $s = 0$ path $\gamma(\ell,s)\big|_{s=0} = \gamma(\ell)$ gives geodesic γ above. Parameter s corresponds to the evolution along field H:

$$\frac{\partial}{\partial s}\gamma(\ell,s) = H(\gamma(\ell,s)). \tag{3.5}$$

In the following lemma we find estimates on the first- and second-order differences in (3.4). Field H will be chosen later to be $H(\ell,s) = A_0(\gamma(\ell,s))$ or $H(\ell,s) = A_\alpha(\gamma(\ell,s))$ correspondingly.

Lemma 3.1 ([8], Lemma 2)**.** *The following estimates on difference operators on metric function are fulfilled*

$$\frac{\rho^2(\gamma(0,\varepsilon),\gamma(1,\varepsilon))-\rho^2(o,x)}{\varepsilon}\le$$

$$\le\int\limits_0^1\frac{\partial}{\partial s}\bigg|_{s=0}\big|\dot\gamma(\ell,s)\big|^2d\ell+\int\limits_0^\varepsilon\int\limits_0^1\left|\frac{\partial^2}{\partial s^2}\big|\dot\gamma(\ell,s)\big|^2\right|d\ell\,ds, \tag{3.6}$$

$$\frac{\rho^2(\gamma(0,\varepsilon),\gamma(1,\varepsilon))+\rho^2(\gamma(0,\varepsilon),\gamma(1,-\varepsilon))-2\rho^2(o,x)}{\varepsilon^2}\le$$

$$\le\int\limits_0^1\frac{\partial^2}{\partial s^2}\bigg|_{s=0}\big|\dot\gamma(\ell,s)\big|^2d\ell+\frac{1}{2}\int\limits_0^\varepsilon\int\limits_0^1\left|\frac{\partial^3}{\partial s^3}\big|\dot\gamma(\ell,s)\big|^2\right|d\ell\,ds, \tag{3.7}$$

where we used notation $\dot\gamma(\ell,s) = \dfrac{\partial}{\partial\ell}\gamma(\ell,s)$.

The right-hand side terms in (3.6), (3.7) *have the following representations in terms of field* H:

$$\frac{\partial}{\partial s}\big|\dot\gamma(\ell,s)\big|^2 = 2\big\langle\dot\gamma,\nabla H[\dot\gamma]\big\rangle, \tag{3.8}$$

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}|\dot\gamma(\ell,\varepsilon)|^2 = \big|\nabla H[\dot\gamma]\big|^2 - \big\langle\dot\gamma,R(H,\dot\gamma)H\big\rangle + \big\langle\dot\gamma,\nabla(\nabla_H H)[\dot\gamma]\big\rangle. \tag{3.9}$$

The third derivative has representation $\dfrac{\partial^3}{\partial s^3}\big|\dot\gamma(\ell,s)\big|^2 = \big\langle\dot\gamma,\mathcal{D}[\dot\gamma]\big\rangle$ *with operator* $\mathcal{D}$ *that depends on the field* H *up to its third-order covariant derivative and on curvature tensor and its covariant derivative.*

Next we use Lemma 3.1 to find weak estimates on operator $\mathcal{L}$. Due to (3.3), this ends the proof of Theorem 3.1.

Lemma 3.2. *Under conditions* (2.1), (2.2), *and* (2.5) *there is constant* K *such that* $\forall\psi\in C^\infty_{0,+}(M\times M)$

$$\int\limits_{M\times M}(\mathcal{L}^*\psi(x,z))\rho^2(x,z)d\sigma(x)d\sigma(z)\le K\int\limits_{M\times M}\psi(x,z)\rho^2(x,z)d\sigma(x)d\sigma(z). \tag{3.10}$$

Proof. Coincides with the proof of [8] (Lemma 3). It is only necessary to choose fields H to be $H(\gamma(\ell,s))=A_0(\gamma(\ell,s))$ or $H(\gamma(\ell,s))=A_\alpha(\gamma(\ell,s))$ for the first- and second-order differences in (3.4).

In [8] (Lemma 3) we used additional factor $(1-\ell)$ in field H, i.e., the choice of field H was $H(\gamma(\ell,s))=(1-\ell)A_0(\gamma(\ell,s))$ or $H(\gamma(\ell,s))=(1-\ell)A_\alpha(\gamma(\ell,s))$, this made point $\gamma(1,s)=z$ to be the same for all s. Therefore, in calculation [8] (ff. (30)–(33)) does not appear additional multiple ℓ and it is a little simpler.

4. Estimates on the continuity with respect to the initial data. Now we apply weak estimates (3.2) to show, similar to [11], that some process on manifold represents a supermartingale. Thus we overcome the difficulties, related with the direct application of the Ito formula arguments (1.6), (1.7).

Recall that process X_t is supermartingale with respect to the flow of σ-algebras $\mathcal{F}_t$ if for all $0\le s\le t$ it is fulfilled $\mathbf{E}(X_t|\mathcal{F}_s)\le X_s$, where $\mathbf{E}(\cdot|\mathcal{F}_s)$ denotes the conditional expectation with respect to σ-algebra $\mathcal{F}_s$.

Theorem 4.1. *Under conditions* (2.1), (2.2), *and* (2.5) *there is an independent of* $U\subseteq M$ *constant* K *such that process*

$$\rho^2(y^U_t(x,z)) - K\int\limits_0^t\rho^2(y^U_s(x,z))ds \tag{4.1}$$

represents an integrable supermartingale with respect to the canonical flow of σ*-algebras* $\mathcal{F}_t$, *related with d-dimensional Wiener process* W^α_t, $\alpha=1,\ldots,d$, *in* (1.1). *Notation* $\rho^2(y^U_t(x,z))$ *means the geodesic distance between first and second components of process* $y^U_t(x,z)$ (3.1) *on the product* $M\times M$.

Moreover, the solution of equation (1.1) *continuously depends on the initial data, i.e., estimate* (1.3) *is true.*

Proof follows lines of proofs of Lemma 4 in [8]. Since we work below with the components of process $y^U_t(x,z)$ and have to make several relevant modifications, we outline its main steps.

Recall, that semigroup P_t^U, generated by localized process $y_t^U(x,z)$ (3.1), preserves the space $C_{0,+}^\infty(M\times M)$ of non-negative C^∞-functions with compact support, so the integrals below are finite. The application of weak estimate (3.10) leads to

$$\forall\,\varphi\in C_{0,+}^\infty(M\times M)\colon\quad \frac{d}{dt}\int\limits_{M\times M}\varphi(x,z)\left\{P_t^U\rho^2(\cdot,\cdot)\right\}(x,z)d\sigma(x)d\sigma(z)=$$

$$=\frac{d}{dt}\int\limits_{M\times M}\left\{[P_t^U]^*\varphi\right\}(x,z)\rho^2(x,z))d\sigma(x)d\sigma(z)=$$

$$=\int\limits_{M\times M}[\mathcal{L}^U]^*\left\{[P_t^U]^*\varphi\right\}(x,z)\rho^2(x,z)d\sigma(x)d\sigma(z)=$$

$$=\int\limits_{M\times M}[\mathcal{L}]^*\left(\zeta^U(x)\zeta^U(z)\left\{[P_t^U]^*\varphi\right\}(x,z)\right)\rho^2(x,z)d\sigma(x)d\sigma(z)\le$$

$$\le K\int\limits_{M\times M}\left\{[P_t^U]^*\varphi\right\}(x,z)\rho^2(x,z)d\sigma(x)d\sigma(z)=$$

$$=K\int\limits_{M\times M}\varphi(x,z)\left\{P_t^U\rho^2(\cdot,\cdot))\right\}(x,z)d\sigma(x)d\sigma(z)$$

where we applied $\mathcal{L}1=0$, used that due to the compactness of support of function $\zeta^U\ge 0$ the integrand $\psi=\zeta^U(x)\zeta^U(z)\left\{[P_t^U]^*\varphi\right\}$ belongs to space $C_{0,+}^\infty(M\times M)$, then applied (3.10) and property $\zeta^U\le 1$.

Therefore for all $\varphi\in C_{0,+}^\infty(M\times M)$ we have estimate

$$\int\limits_{M\times M}\varphi(x,z)\left\{P_t^U\rho^2(\cdot,\cdot)\right\}(x,z)d\sigma(x)d\sigma(z)\le$$

$$\le\int\limits_{M\times M}\varphi(x,z)\left(\rho^2(x,z))+K\int\limits_0^t\left\{P_s^U\rho^2(\cdot,\cdot))\right\}(x,z)ds\right)d\sigma(x)d\sigma(z)$$

and, removing φ, conclude its pointwise consequence

$$\left\{P_t^U\rho^2(\cdot,\cdot)\right\}(x,z)\le\rho^2(x,z))+K\int\limits_0^t\left\{P_s^U\rho^2(\cdot,\cdot)\right\}(x,z)ds. \tag{4.2}$$

The Markov property of process $y_t^U(x,z)$ implies for its semigroup P_t^U that

$$(P_t^Uf)(y_s^U(x,z))=\mathbf{E}\big(f(y_{t+s}^U(x,z))\,\big|\,\mathcal{F}_s\big),\quad t,s\ge 0, \tag{4.3}$$

which permits to substitute instead of x, z initial data $y_\tau^U(x,z)$ in (4.2). From (4.3) we have

$$\mathbf{E}\big(\rho^2(y_{t+\tau}^U(x,z))\,\big|\,\mathcal{F}_\tau\big)=(P_t^U\rho^2(\cdot,\cdot))(y_\tau^U(x,z))\le$$

$$\le \rho^2(y_\tau^U(x,z)) + K\int\limits_0^t \left\{P_s^U \rho^2(\cdot,\cdot)\right\}(y_\tau^U(x,z))ds =$$

$$= \rho^2(y_\tau^U(x,z)) + K\,\mathbf{E}\left(\int\limits_\tau^{t+\tau} \rho^2(y_s^U(x,z))ds \,\middle|\, \mathcal{F}_\tau\right), \tag{4.4}$$

which means that process (4.1) is supermartingale. Indeed, the supermartingale property

$$\mathbf{E}\left(\rho^2(y_{t+\tau}^U(x,z)) - K\int\limits_0^{t+\tau} \rho^2(y_s^U(x,z))ds \,|\, \mathcal{F}_\tau\right) \le$$

$$\le \rho^2(y_\tau^U(x,z)) - K\int\limits_0^\tau \rho^2(y_s^U(x,z))ds$$

coincides with (4.4). The integrability of process (4.1) follows from the compactness of the closure of set $\{x\colon \zeta^U(x) > 0\}$.

Next suppose that initial data $x, z \in U$. Introduce stopping time

$$\tau^U(\omega) = \inf\left\{t \ge 0\colon y_t^U(x,z) \notin U \times U\right\}.$$

The Doob – Meyer free choice theorem, e.g. [12] (Ch. VI, § 2), permits to substitute any finite stopping times S, T such that $0 \le S \le T$ into the supermartingale property $\mathbf{E}(X_T|\mathcal{F}_S) \le X_S$. Let's apply it with $S = 0$ and $T = t \wedge \tau^U$ to supermartingale (4.1). Due to $\mathbf{E}(\cdot|\mathcal{F}_0) = \mathbf{E}(\cdot)$ we have

$$m_t = \mathbf{E}\rho^2(y_{t\wedge\tau^U}^U(x,z)) \le \rho^2(x,z) + K\mathbf{E}\int\limits_0^{t\wedge\tau^U} \rho^2(y_s^U(x,z))ds \le$$

$$\le m_0 + K\mathbf{E}\int\limits_0^t \rho^2(y_{s\wedge\tau^U}^U(x,z))ds = m_0 + K\int\limits_0^t m_s ds,$$

where $y_{s\wedge\tau^U}^U(x,z) = y_{\tau^U}^U(x,z)$ for $s \ge \tau^U$ is a stopped process on the boundary of U and we enlarged the upper limit of integral.

From Gronwall – Bellmann inequality we conclude

$$\mathbf{E}\rho^2(y_{t\wedge\tau^U}^U(x,z))) \le e^{Kt}\rho^2(x,z). \tag{4.5}$$

Let U_n denote the open ball at point o with radius n, then for sufficiently large n points $x, z \in U_n$. Consider measurable random set $V_n(t) = \{\omega\colon \forall s \in [0,t]\ y_t^x(\omega) \in U_n$ and $y_t^z(\omega) \in U_n\}$ that corresponds to paths of processes $y_t^x(\omega)$, $y_t^z(\omega)$ (1.1), staying inside of set U_n till time t. Then $\left(y_t^x(\omega), y_t^z(\omega)\right) = y_{t\wedge\tau^{U_n}}^{U_n}(x,z,\omega)$ for all $\omega \in V_n(t)$ and (4.5) leads to

$$\mathbf{E}\,1_{V_n(t)}\rho^2(y_t^x, y_t^z) = \mathbf{E}\,1_{V_n(t)}\rho^2(y_{t\wedge\tau^{U_n}}^{U_n}(x,z)) \le$$

$$\le \mathbf{E}\rho^2(y_{t\wedge\tau^{U_n}}^{U_n}(x,z)) \le e^{Kt}\rho^2(x,z), \tag{4.6}$$

with characteristic function $1_{V_n(t)}$ of set $V_n(t)$.

Due to the non-explosion $\lim_{n\to\infty} \tau^{U_n}(\omega) = \infty$ [8], for a.e. ω both paths $y_t^x(\omega), y_t^z(\omega)$ sooner or later completely lie in all sets U_n for $n \geq n_0$ with sufficiently large n_0. Therefore sequence $V_n(t)$ is increasing to the full probability space, i.e., lower limit $\underline{\lim}_{n\to\infty} 1_{V_n(t)}(\omega) = 1$ a.e. The application of Fatoux lemma $\Big($i.e., that for $f_n \geq 0$ the lower limits fulfill $\int \underline{\lim}_{n\to\infty} f_n d\mu \leq \underline{\lim}_{n\to\infty} \int f_n d\mu \Big)$ to the left-hand side of (4.6) leads to the statement

$$\mathbf{E}\,\rho^2(y_t^x, y_t^z) \leq \lim_{n\to\infty} \mathbf{E}\rho^2(y^{U_n}_{t\wedge\tau^{U_n}}(x,z)) \leq e^{Kt}\rho^2(x,z). \tag{4.7}$$

The theorem is proved.

In the following theorem we generalize Theorem 4.1 to the polynomials of metric function. Remark that the convex function of supermartingale should not be a supermartingale again, so the use of coercitivity and dissipativity conditions (2.1), (2.2) is essential for existence of appropriate constant K_Q in (4.9).

Theorem 4.2. *Let Q be a nonnegative monotone polynomial function on half-line $\mathbb{R}_+$ such that*

$$\exists\, C \quad \forall\, z \geq 0 \quad z\,Q'(z) \leq C\,Q(z), \qquad z\,\big|Q''(z)\big| \leq C\,Q'(z). \tag{4.8}$$

Under conditions (2.1), (2.2) *and* (2.5) *there is constant K_Q such that uniformly on vicinity U the process*

$$Q\big(\rho^2(y_t^U(x,z))\big) - K_Q \int_0^t Q\big(\rho^2(y_s^U(x,z))\big)ds \tag{4.9}$$

is an integrable supermartingale.

Moreover, a unique solution y_t^x to problem (1.1) *fulfills the estimate on the continuity with respect to the initial data*

$$\mathbf{E}Q\big(\rho^2(y_t^x, y_t^z)\big) \leq e^{K_Q t} Q\big(\rho^2(x,z)\big). \tag{4.10}$$

Proof is done in analogue to [8] (Theorem 5).

5. Absolute-continuity approach to the first-order regularity with respect to the initial data. Consider arbitrary smooth path $h \in C^\infty([a,b], M)$ that starts at point $x = h(a)$. To obtain the equation on the first variation $y_t^{(1)}(x)\big[h'(a)\big] = \frac{d}{du}\Big|_{u=0} y_t^{h(u)}$ let us formally differentiate (1.2)

$$\Big\langle \nabla f(y_t^x), y_t^{(1)}(x)\big[h'(a)\big] \Big\rangle =$$

$$= \big\langle \nabla f(x), h'(a) \big\rangle + \sum_\alpha \int_0^t \Big\langle \nabla(A_\alpha f)(y_s^x),\ y_s^{(1)}(x)\big[h'(a)\big] \Big\rangle \delta W_s^\alpha +$$

$$+ \int_0^t \Big\langle \nabla(A_0 f)(y_s^x),\ y_s^{(1)}(x)\big[h'(a)\big] \Big\rangle ds. \tag{5.1}$$

This leads to the equation on the first-order variation in local coordinates e.g. [13, 14]

$$\delta\big[y_t^{(1)}(x)\big]_j^m = -\Gamma_{p\ q}^m(y_t^x)\big[y_t^{(1)}(x)\big]_j^p \delta(y_t^x)^q +$$

$$+\sum_\alpha \nabla_p A_\alpha^m(y_t^x)\big[y_t^{(1)}(x)\big]_j^p \delta W_t^\alpha + \nabla_p A_0^m(y_t^x)\big[y_t^{(1)}(x)\big]_j^p dt \tag{5.2}$$

with initial data $y_0^{(1)} = I$.

In next theorem we give sufficient conditions for the solvability of equation (5.2), see also e.g. [13] (Theorem 15).

Theorem 5.1. *Suppose that conditions* (2.1), (2.2), *and* (2.5) *are fulfilled. Then equation* (5.2) *has unique solution, i.e., exists a continuous adapted process* $y_t^{(1)}(x)[h']$ *with values in* $T_{y_t^x}M$, *such that for any* $f \in C_0^\infty(M)$ *and* $h' \in T_xM$ *relation* (5.1) *is true.*

In particular,

$$\forall\, p \geq 1 \quad \exists\, K_p \quad \textit{such that} \quad \mathbf{E}\big\|y_t^{(1)}(x)\big\|_{T_{y_t^x}\otimes T_x^*M}^p \leq e^{K_p t}. \tag{5.3}$$

Proof. Since equation on the first variation is formally calculated as derivative of (1.1), it also has the following equivalent form to (5.2) (see [13], (3.1)):

$$\delta\big(\big[y^{(1)}(x)_j^m\big]\big) = \sum_\alpha \frac{\partial A_\alpha^m(y_t^x)}{\partial y^p}\big[y_t^{(1)}(x)\big]_j^p \delta W_t^\alpha + \frac{\partial A_0^m(y_t^x)}{\partial y^p}\big[y_t^{(1)}(x)\big]_j^p dt. \tag{5.4}$$

Therefore it represents an equation with the locally Lipschitz coefficients. Standard results about the solvability of finite-dimensional diffusion equations lead to the local existence and uniqueness of its solutions till the first explosion time, e.g. [2, 4, 6, 11].

The non-explosion of process $y_t^{(1)}(x)$ follows from the following representation of the local differential of its norm

$$d\|y_t^{(1)}(x)\|^2 = 2\Big\langle y_t^{(1)}(x), \nabla_\ell^y A_\alpha\left[\big[y_t^{(1)}(x)\big]^\ell\right]\Big\rangle dW^\alpha +$$

$$+\Bigg\{ 2\Big\langle y_t^{(1)}(x), \nabla_\ell^y \widetilde{A_0}\left[\big[y_t^{(1)}(x)\big]^\ell\right]\Big\rangle + \sum_{\alpha=1}^d \Big\|\nabla A_\alpha\big[y_t^{(1)}(x)\big]\Big\|^2 -$$

$$-\sum_{\alpha=1}^d \Big\langle R(A_\alpha, y_t^{(1)}(x))A_\alpha, y_t^{(1)}(x)\Big\rangle\Bigg\} dt,$$

proved as a base of recurrence (for $\gamma = \varnothing$) in Lemma 13 [13] (see also [13], (4.28) for $i = 1$).

Therefore the dissipativity condition (2.2) arises in the second line. Due to the initial data $y_0^{(1)}(x) = \dfrac{\partial x}{\partial x} = Id$ and Gronwall – Bellmann inequality, it leads to the non-explosion estimate (5.3), i.e., to the existence and uniqueness of solution $y_t^{(1)}(x)$ to (5.2), (5.4) for all $t \geq 0$.

The theorem is proved.

In the following theorem we apply the theory of absolute continuous functions to show the first-order regularity of diffusion process.

Theorem 5.2. *Under conditions* (2.1), (2.2), *and* (2.5) *process* y_t^x *is differentiable with respect to the initial data. For any Lipschitz continuous path* $h \in \mathrm{Lip}([a,b], M)$ *its derivative* $\dfrac{dy_t^{h(z)}}{dz}$ *is represented by solution of the first-order variational equation* (5.2)

$$\frac{dy_t^{h(z)}}{dz} = y_t^{(1)}(h(z))\big[h'(z)\big]$$

and a.e. integral relation (2.7) *is fulfilled.*

Proof. Let us prove relation (2.7) for functions f with sufficiently small support, then the use of the decomposition of identity guarantees (2.7) for arbitrary $f \in C_0^\infty(M)$.

The main idea of proof is following: for any small vicinity $U \subset M$ we are going to construct the globally defined functions $\theta^i \in C(M)$, $i = 1, \ldots, \dim M$, such that

1) the superposition with diffusion $\theta^i \circ y_t^x$ is regular, i.e., exists $\dfrac{\partial \theta^i(y_t^x)}{\partial x}$ (Step 1);

2) any function $f \in C_0^\infty(M)$ with compact support in U has an unique representation in terms on θ^i (Step 2), i.e.,

$$\exists \widetilde{f} \in C^\infty(\mathbb{R}^{\dim M}, \mathbb{R}) \quad \text{such that} \quad \forall\, x \in U \quad f(x) = \widetilde{f}(\theta^1(x), \ldots, \theta^{\dim M}(x)).$$

These properties guarantee that for any $f \in C_0^\infty(U)$ expression $f(y_t^x) = \widetilde{f}\big(\theta^1(y_t^x), \ldots, \theta^{\dim M}(y_t^x)\big)$ is again regular with respect to the initial data.

Finally, in Step 3 we will use the last property to derive the equation on derivative $\dfrac{\partial y_t^x}{\partial x}$.

Step 1. Construction of special coordinate system and use of continuity estimates (4.10) to guarantee the existence of derivative with respect to the initial data.

First note that for any point $o \in M$ there is a sufficiently small vicinity $U = U(o) \ni o$ and points outside of this vicinity $o_i = o_i(o) \notin U$, $i = 1, \ldots, \dim M$, such that they generate the smooth local coordinate's mapping in U

$$\overline{\theta}(x) = \big(\theta^i(x)\big)_{i=1}^{\dim M} : U \to \mathbb{R}^{\dim M} \quad \text{by rule} \quad \theta^i(x) = \rho(o_i, x).$$

Recall that above $\theta^i(x) = \rho(o_i, x)$ denotes the shortest geodesic distance from $x \in U$ to point o_i, $i = 1, \ldots, \dim M$.

Vicinity U must be also chosen so that for any point $x \in U$ there is no point $z \in U$ such that it has the same coordinates $\overline{\theta}(x) = \overline{\theta}(z)$. Last assumption actually means that the points $o_i(o)$ are sufficiently far from U, so that the "phantom" images of U

$$Ph(U) = \big\{ z \notin U : \overline{\theta}(z) = \overline{\theta}(x) \ \text{ for some } \ x \in U \big\}$$

do not intersect with U. Moreover, by varying the size of $U(o)$ and points $o_i(o) \notin U(o)$ we can guarantee that "phantom" images of U are far from U:

$$\exists\, \varepsilon > 0 \quad \forall\, o \in M \quad \mathrm{dist}\big(U(o), Ph(U(o))\big) > 2\varepsilon. \tag{5.5}$$

Next, since for sufficiently small U the coordinate mapping $\overline{\theta} \colon U \to \mathbb{R}^{\dim M}$ becomes bijection with continuous inverse, the set $\overline{\theta}(U) = \big\{ \overline{\theta}(x) \colon x \in U \big\} \subset \mathbb{R}^{\dim M}$, being a preimage of open set U, is open too. Moreover, due to the role of parameter ε (5.5), i.e., the absence of equidistant points, the mapping $\overline{\theta}$ is C^∞-smooth in the ε-vicinity of U with C^∞-smooth inverse.

Finally remark that $\theta^i(x)$ are globally defined continuous functions on the manifold M, but only in the ε-vicinity of U they form the coordinate system. In particular, due to the triangle inequality

$$\big|\rho(o_i,x)-\rho(o_i,z)\big|\le\rho(x,z), \tag{5.6}$$

functions θ^i are globally Lipschitz continuous with constant 1.

Introduce processes

$$\theta^i(y_t^x)=\rho(o_i,y_t^x),$$

then from (4.10) and (5.6) we have

$$\forall\, p\ge 1,\quad T>0:\quad \sup_{t\in[0,T]}\mathbf{E}\big|\theta^i(y_t^x)-\theta^i(y_t^z)\big|^p\le$$

$$\le\sup_{t\in[0,T]}\mathbf{E}\big[\rho(y_t^x,y_t^z)\big]^p\le e^{K_pT}\rho^p(x,z).$$

Therefore path $[a,b]\in z\to\theta^i(y_t^{h(z)})\in L^\infty\big([0,T],L^p(\Omega,\mathcal{W})\big)$ is Lipschitz continuous for Lipschitz continuous $h\in\mathrm{Lip}\,([a,b],M)$:

$$\forall\, c,\ d\in[a,b]:\quad \sup_{t\in[0,T]}\mathbf{E}|\theta^i\big(y_t^{h(c)}\big)-\theta^i\big(y_t^{h(d)}\big)|^p\le|c-d|e^{K_pT}\|h'\|_{L^\infty([a,b],TM)}.$$

By theory of absolute continuous functions there exists derivative

$$\frac{d\theta^i(y_t^{h(z)})}{dz}\in L^\infty\big([a,b]\times[0,T],L^p(\Omega,\mathcal{W})\big)$$

with Lipschitzness constant

$$\sup_{z\in[a,b],t\in[0,T]}\mathbf{E}\left\|\frac{d\theta^i(y_t^{h(z)})}{dz}\right\|^p_{T_{y_t^{h(z)}}M\otimes T^*_{h(z)}M}\le e^{K_pT}\|h'\|^p_{L^\infty([a,b],TM)} \tag{5.7}$$

and we have a.e. relation

$$\theta^i\big(y_t^{h(b)}\big)-\theta^i\big(y_t^{h(a)}\big)=\int\limits_a^b\frac{d\theta^i(y_t^{h(z)})}{dz}dz. \tag{5.8}$$

Above by $\mathcal{W}$ we denoted Wiener measure, related with process $\{W_t^\alpha\}_\alpha$.

Step 2. Construction of unique $\mathbb{R}^{\dim M}$-representations of functions in small coordinate vicinities.

Next note that any function $f\in C_0^\infty(U)$ with compact support in U there is a smooth function $\widetilde{f}\in C_0^\infty(\mathbb{R}^{\dim M})$ that provides its unique coordinate representation in terms of coordinates $\overline{\theta}(x)$

$$f(x)=1_U(x)\widetilde{f}(\overline{\theta}(x)), \tag{5.9}$$

where $1_U(x)$ denotes the characteristic function of U.

At the first step, function $\widetilde{f}$ is defined as the coordinate version of f in coordinates $\overline{\theta}$. Then it is continued to all $\mathbb{R}^{\dim M}$ by zero outside of the open set $\overline{\theta}(U)$. The characteristic $1_U(x)$ is added to avoid the "phantom" copies of function f outside of vicinity U, which appear at the equidistant points z such that $\theta^i(x) = \rho(o_i, x) = \rho(o_i, z) = \theta^i(z)$, $i = 1, \ldots, \dim M$. Remark also that property (5.9) is not true for all functions f on M because $\overline{\theta}(x)$ can not form a global coordinate system on M.

However, the presence of factor $1_U(x)$ in (5.9) does not influence further calculations because the compact support of f completely lies in the open set U. For example, the differential operations do not feel the factor $1_U(x)$

$$(Af)(x) = (\nabla_A f)(x) = 1_U(x) \sum_{j=1}^{\dim M} \partial_j \widetilde{f}(\overline{\theta}(x)) \nabla_A \theta^j(x) =$$

$$= 1_U(x) \sum_{j=1}^{\dim M} (\widetilde{A}^j \partial_j \widetilde{f})(\overline{\theta}(x)) = 1_U(x)\, (\widetilde{A}\widetilde{f})\big(\overline{\theta}(x)\big)$$

with local coordinates $\widetilde{A}^j(\overline{\theta}(x)) = A\theta^j(x)$ of vector field A in the vicinity U.

Remark that each function $\widetilde{A}^j$ can be uniquely extended to C_0^∞-function over all $\mathbb{R}^{\dim M}$: for points $\overline{\theta}(x) \in \mathbb{R}^{\dim M}$ with $x \in M$ such that dist$(x, U) \leq \varepsilon$ it is defined by formula

$$\widetilde{A}^j\big(\overline{\theta}(x)\big) = \chi_\varepsilon\big(\text{dist}(x, U)\big) A\theta^j(x)$$

and by zero at all other points of $\mathbb{R}^{\dim M}$. A fixed and independent on $o \in M$ function $\chi_\varepsilon \in C^\infty(\mathbb{R}_+, [0, 1])$ is such that $\chi_\varepsilon(0) = 1$ and $\chi_\varepsilon(\lambda) = 0$, $\lambda \geq \varepsilon$, parameter ε appeared in (5.5). In other words, we take the coordinates $\widetilde{A}^j$ in the image of ε-vicinity of U, leave them unchanged in $\overline{\theta}(U)$ and crop them to zero outside of $\overline{\theta}$-image of the ε-vicinity of U. Since the mapping $\overline{\theta}$ is C^∞-smooth in the ε-vicinity of U with C^∞-smooth inverse, we obtain C_0^∞-regularity of $\widetilde{A}^j$. However, due to $f \in C_0^\infty(U)$ the function $Af \in C_0^\infty(U)$, i.e., the values of field A outside of U do not play a role.

Remark also that the representation (5.9) is more comfortable than, for example, the use of embeddings of manifold M into Euclidean spaces $M \subset \mathbb{R}^n$ of higher dimensions $n > \dim M$. In this case one should use the global coordinates of $\mathbb{R}^n$ instead of the local coordinates of M, e.g. [6, 11] and references therein. In particular, such approach leads to complicate work with the continuation of coefficients of equation from embedded submanifold $M \subset \mathbb{R}^n$ to $\mathbb{R}^n$ and forces to enter additional projectors from $\mathbb{R}^n$ to M for embedded $\mathbb{R}^n$-versions of (1.1).

One more advantage is that for functions inside of U we have their unique representations $\widetilde{f}$ in terms of local coordinates $\theta^i(x)$, i.e., a unique function $\widetilde{f}$ on $\mathbb{R}^{\dim M}$. This property will permit us below to make all calculations for local functions $f \in C_0^\infty(U)$ in linear coordinate space $\mathbb{R}^{\dim M}$.

Step 3. Derivation of stochastic equation for derivative $\dfrac{d\overline{\theta}(y_s^{h(u)})}{du}$ and its relation with the first-order variation process $y_s^{(1)}(h(u))$.

Since the superposition of smooth finite function and Lipschitz continuous function is also Lipschitz continuous, we have from (5.8)

$$f\big(y_t^{h(b)}\big) - f\big(y_t^{h(a)}\big) = 1_U\big(y_t^{h(z)}\big) \widetilde{f}\big(\overline{\theta}(y_t^{h(z)})\big) \Big|_{z=a}^{z=b} =$$

$$= \int_a^b 1_U\big(y_t^{h(z)}\big) \sum_{j=1}^{\dim M} \partial_j \widetilde{f}\big(\overline{\theta}(y_t^{h(z)})\big) \frac{d\theta^j(y_t^{h(z)})}{dz} dz. \tag{5.10}$$

From another side, the definition of solution y_t^x (1.2) leads to the representation for difference

$$f\big(y_t^{h(b)}\big) - f\big(y_t^{h(a)}\big) = f\big(h(b)\big) - f\big(h(a)\big) +$$

$$+ \sum_\alpha \int_0^t \Big[(A_\alpha f)\big(y_s^{h(b)}\big) - (A_\alpha f)\big(y_s^{h(a)}\big)\Big]\, \delta W_s^\alpha +$$

$$+ \int_0^t \Big[(A_0 f)\big(y_s^{h(b)}\big) - (A_0 f)\big(y_s^{h(a)}\big)\Big]\, ds =$$

$$= 1_U(h(b))\widetilde{f}\big(\overline{\theta}(h(b))\big) - 1_U(h(a))\widetilde{f}\big(\overline{\theta}(h(b))\big) +$$

$$+ \sum_\alpha \int_0^t \Big[1_U(y_s^{h(b)})\,\big(\widetilde{A_\alpha}\widetilde{f}\big)\big(\overline{\theta}(y_s^{h(b)})\big) - 1_U\big(y_s^{h(a)}\big)\,\big(\widetilde{A_\alpha}\widetilde{f}\big)\big(\overline{\theta}(y_s^{h(a)})\big)\Big]\, \delta W_s^\alpha +$$

$$+ \int_0^t \Big[1_U\big(y_s^{h(b)}\big)\,\big(\widetilde{A_0}\widetilde{f}\big)\big(\overline{\theta}(y_s^{h(b)})\big) - 1_U\big(y_s^{h(a)}\big)\,\big(\widetilde{A_0}\widetilde{f}\big)\big(\overline{\theta}(y_s^{h(a)})\big)\Big]\, ds. \qquad (5.11)$$

Again applying that the superposition of smooth function $\widetilde{A}\widetilde{f} \in C_0^\infty(\mathbb{R}^d)$ and Lipschitz continuous map $[a,b] \to \theta^i\big(y_s^{h(z)}\big)$ is also Lipschitz continuous, we obtain from (5.10) and (5.11) that

$$\int_a^b 1_U\big(y_t^{h(z)}\big) \sum_{j=1}^{\dim M} \partial_j \widetilde{f}\big(\overline{\theta}(y_t^{h(z)})\big)\, \frac{d\theta^j\big(y_t^{h(z)}\big)}{dz} dz =$$

$$= \int_a^b 1_U(h(z)) \sum_{j=1}^{\dim M} \partial_j \widetilde{f}\big(\overline{\theta}(h(z))\big)\, \frac{d\theta^j(h(z))}{dz} dz +$$

$$+ \sum_\alpha \int_0^t \left[\int_a^b 1_U\big(y_s^{h(z)}\big) \sum_{j=1}^{\dim M} \big(\partial_j \widetilde{A_\alpha}\widetilde{f}\big)\big(\overline{\theta}(y_s^{h(z)})\big)\, \frac{d\theta^j\big(y_s^{h(z)}\big)}{dz} dz\right] \delta W_s^\alpha +$$

$$+ \int_0^t \left[\int_a^b 1_U\big(y_s^{h(z)}\big) \sum_{j=1}^{\dim M} \big(\partial_j \widetilde{A_0}\widetilde{f}\big)\big(\overline{\theta}(y_s^{h(z)})\big)\, \frac{d\theta^j\big(y_s^{h(z)}\big)}{dz} dz\right] ds.$$

Due to (5.3) and (5.7) the terms under integrals above are in $L^\infty\big([a,b] \times [0,T], L^p(\Omega, \mathcal{W})\big)$, $p \geq 1, T > 0$. So the order of integrals $\int_a^b$ and $\int_0^t$ can be changed. As $h \in \mathrm{Lip}([a,b], M)$ and $[a,b]$ were arbitrary, the integrands under $\int_a^b$ must coincide: for almost all $z \in [a,b]$

$$1_U\big(y_t^{h(z)}\big)\sum_{j=1}^{\dim M}\partial_j\widetilde{f}\big(\overline{\theta}(y_t^{h(z)})\big)\,\frac{d\theta^j(y_t^{h(z)})}{dz}=$$

$$=1_U(h(z))\sum_{j=1}^{\dim M}\partial_j\widetilde{f}\big(\overline{\theta}(h(z))\big)\,\frac{d\theta^j(h(z))}{dz}+$$

$$+\sum_\alpha\int_0^t\left[1_U\big(y_s^{h(z)}\big)\sum_{j=1}^{\dim M}\big(\partial_j\widetilde{A_\alpha f}\big)\big(\overline{\theta}(y_s^{h(z)})\big)\,\frac{d\theta^j(y_s^{h(z)})}{dz}\right]\delta W_s^\alpha+$$

$$+\int_0^t\left[1_U\big(y_s^{h(z)}\big)\sum_{j=1}^{\dim M}\big(\partial_j\widetilde{A_0 f}\big)\big(\overline{\theta}(y_s^{h(z)})\big)\,\frac{d\theta^j(y_s^{h(z)})}{dz}\right]ds.$$

Turning back to the invariant notations and fields on M we obtain

$$\left\langle\nabla f\big(y_t^{h(z)}\big),\frac{dy_t^{h(z)}}{dz}\right\rangle=\left\langle\nabla f(h(z)),\frac{dh(z)}{dz}\right\rangle+$$

$$+\sum_\alpha\int_0^t\left\langle\nabla(A_\alpha f)\big(y_s^{h(z)}\big),\frac{dy_s^{h(z)}}{dz}\right\rangle\delta W_s^\alpha+\int_0^t\left\langle\nabla(A_0 f)\big(y_s^{h(z)}\big),\frac{dy_s^{h(z)}}{dz}\right\rangle ds. \tag{5.12}$$

Here we used that when $y_s^{h(z)}\in U$ all terms with index j represent coordinates of corresponding tensor-invariant objects. From another side, when $y_s^{h(z)}\notin U$ terms with index j are no more coordinates, but as the multiple $1_U(y_s^{h(z)})=0$ and supports of f, $A_\alpha f$, $A_0 f$ lie in U, these terms are also invariant, being defined by zero outside of U.

Relation (5.12) is a further advantage of relation (5.9): it is fulfilled for the process y_t^x that may many times enter and leave vicinity U. Therefore we do not need to restrict the consideration till the first exit times and use local arguments.

Finally notice that equations (5.12) and (5.1) have the same structure. Due to the coinciding initial data $\dfrac{d\theta^j(h(z))}{dz}=\big[h'(z)\big]^j$ and the uniqueness of solutions for equation (5.1), the first variation coincides with the derivative with respect to the initial data

$$y_t^{(1)}(h(z))\big[h'(z)\big]=\frac{dy_t^{h(z)}}{dz}.$$

After substitution of this relation into (5.10) we come to (2.7).

The theorem is proved.

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