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INTEGRAL GROUP RING OF RUDVALIS SIMPLE GROUP*

ІНТЕГРАЛЬНЕ ГРУПОВЕ КІЛЬЦЕ ПРОСТОЇ ГРУПИ РУДВАЛІСА

Using the Luthar – Passi method, we investigate the classical Zassenhaus conjecture for the normalized unit group of the integral group ring of the Rudvalis sporadic simple group Ru. As a consequence, for this group we confirm Kimmerle's conjecture on prime graphs.

За допомогою методу Лутара – Пассі досліджено класичну гіпотезу Цассенхауза для нормованої мультиплікативної групи цілочисельного групового кільця спорадичної простої групи Рудваліса Ru. Як наслідок, для цієї групи підтверджено гіпотезу Кіммерле щодо графів простих чисел.

1. Introduction, conjectures and main results. Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$ of a finite group G. A long-standing conjecture of H. Zassenhaus **(ZC)** says that every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element in G (see [1]).

For finite simple groups the main tool for the investigation of the Zassenhaus conjecture is the Luthar – Passi method, introduced in [2] to solve it for A_5 and then applied in [3] for the case of S_5. Later M. Hertweck in [4] extended the Luthar – Passi method and applied it for the investigation of the Zassenhaus conjecture for $PSL(2, p^n)$. The Luthar – Passi method proved to be useful for groups containing non-trivial normal subgroups as well. For some recent results we refer to [4 – 9]. Also, some related properties and some weakened variations of the Zassenhaus conjecture can be found in [3, 10, 11].

First of all, we need to introduce some notation. By $\#(G)$ we denote the set of all primes dividing the order of G. The Gruenberg – Kegel graph (or the prime graph) of G is the graph $\pi(G)$ with vertices labeled by the primes in $\#(G)$ and with an edge from p to q if there is an element of order pq in the group G. In [12] W. Kimmerle proposed the following weakened variation of the Zassenhaus conjecture:

(KC) If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

In particular, in the same paper W. Kimmerle verified that **(KC)** holds for finite Frobenius and solvable groups. We remark that with respect to the so-called p-version of the Zassenhaus conjecture the investigation of Frobenius groups was completed by M. Hertweck and the first author in [13]. In [6, 14 – 18] **(KC)** was confirmed for the Mathieu simple groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24} and the sporadic Janko simple groups J_1, J_2, and J_3.

Here we continue these investigations for the Rudvalis simple group Ru. Although using the Luthar – Passi method we cannot prove the rational conjugacy for torsion units of $V(\mathbb{Z}\,\mathrm{Ru})$, our main result gives a lot of information on partial augmentations of these units. In particular, we confirm the Kimmerle's conjecture for this group.

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Let $G = \mathrm{Ru}$. It is well known (see [19]) that $|G| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ and $\exp(G) = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 29$. Let

$$\mathcal{C} = \Big\{C_1,\ C_{2a},\ C_{2b},\ C_{3a},\ C_{4a},\ C_{4b},\ C_{4c},\ C_{4d},\ C_{5a},\ C_{5b},\ C_{6a},\ C_{7a},\ C_{8a},\ C_{8b},\ C_{8c},\ C_{10a},\ C_{10b},\ C_{12a},\ C_{12b},\ C_{13a},\ C_{14a},\ C_{14b},\ C_{14c},\ C_{15a},\ C_{16a},\ C_{16b},\ C_{20a},\ C_{20b},\ C_{20c},\ C_{24a},\ C_{24b},\ C_{26a},\ C_{26b},\ C_{26c},\ C_{29a},\ C_{29b}\Big\}$$

be the collection of all conjugacy classes of Ru, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order k. Denote by $\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$ the partial augmentation of u with respect to C_{nt}. From the Berman–Higman Theorem (see [20] and Ch. 5 [21, p. 102]) one knows that $\nu_1 = \alpha_1 = 0$ and

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1. \tag{1}$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of the conjugacy class C_{nt}.

Our main result is the following theorem:

Theorem 1. *Let G denote the Rudvalis sporadic simple group* Ru. *Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$ and let*

$$\mathfrak{P}(u) = \Big(\nu_{2a},\ \nu_{2b},\ \nu_{3a},\ \nu_{4a},\ \nu_{4b},\ \nu_{4c},\ \nu_{4d},\ \nu_{5a},\ \nu_{5b},\ \nu_{6a},\ \nu_{7a},\ \nu_{8a},\ \nu_{8b},\ \nu_{8c},\ \nu_{10a},\ \nu_{10b},\ \nu_{12a}, \nu_{12b},\ \nu_{13a},\ \nu_{14a},\ \nu_{14b},\ \nu_{14c},\ \nu_{15a},\ \nu_{16a},\ \nu_{16b}, \nu_{20a},\ \nu_{20b},\ \nu_{20c},\ \nu_{24a},\ \nu_{24b},\ \nu_{26a},\ \nu_{26b},\ \nu_{26c},\ \nu_{29a},\ \nu_{29b}\Big) \in \mathbb{Z}^{35}$$

be the tuple of partial augmentations of u. The following properties hold.

(i) *If* $|u| \notin \{28,\ 30,\ 40,\ 48,\ 52,\ 56,\ 60,\ 80,\ 104,\ 112,\ 120,\ 208,\ 240\}$, *then* $|u|$ *coincides with the order of some element $g \in G$. Equivalently, there is no elements of orders* 21, 35, 39, 58, 65, 87, 91, 145, 203 *and* 377 *in* $V(\mathbb{Z}G)$.

(ii) *If* $|u| \in \{3,\ 7,\ 13\}$, *then u is rationally conjugate to some $g \in G$.*

(iii) *If* $|u| = 2$, *the tuple of the partial augmentations of u belongs to the set*

$$\big\{\mathfrak{P}(u) \mid \nu_{2a} + \nu_{2b} = 1,\ -10 \le \nu_{2a} \le 11,\ \nu_{kx} = 0,\ kx \notin \{2a, 2b\}\big\}.$$

(iv) *If* $|u| = 5$, *the tuple of the partial augmentations of u belongs to the set*

$$\big\{\mathfrak{P}(u) \mid \nu_{5a} + \nu_{5b} = 1,\ -1 \le \nu_{5a} \le 6,\ \nu_{kx} = 0,\ kx \notin \{5a, 5b\}\big\}.$$

(v) *If* $|u| = 29$, *the tuple of the partial augmentations of u belongs to the set*

$$\big\{\mathfrak{P}(u) \mid \nu_{29a} + \nu_{29b} = 1,\ -4 \le \nu_{29a} \le 5,\ \nu_{kx} = 0,\ kx \notin \{29a, 29b\}\big\}.$$

As an immediate consequence of part (i) of the Theorem 1 we obtain the following corollary:

***Corollary* 1.** *If* $G = \mathrm{Ru}$ *then* $\pi(G) = \pi(V(\mathbb{Z}G))$.

2. Preliminaries. The following result is a reformulation of the Zassenhaus conjecture in terms of vanishing of partial augmentations of torsion units.

Proposition 1 (see [2] and Theorem 2.5 in [22])**.** *Let* $u \in V(\mathbb{Z}G)$ *be of order* k. *Then* u *is conjugate in* $\mathbb{Q}G$ *to an element* $g \in G$ *if and only if for each* d *dividing* k *there is precisely one conjugacy class* C *with partial augmentation* $\varepsilon_C(u^d) \neq 0$.

The next result now yield that several partial augmentations are zero.

Proposition 2 (see [7], Proposition 3.1; [4], Proposition 2.2)**.** *Let* G *be a finite group and let* u *be a torsion unit in* $V(\mathbb{Z}G)$. *If* x *is an element of* G *whose* p*-part, for some prime* p, *has order strictly greater than the order of the* p*-part of* u, *then* $\varepsilon_x(u) = 0$.

The key restriction on partial augmentations is given by the following result that is the cornerstone of the Luthar – Passi method.

Proposition 3 (see [2, 4])**.** *Let either* $p = 0$ *or* p *a prime divisor of* $|G|$. *Suppose that* $u \in V(\mathbb{Z}G)$ *has finite order* k *and assume* k *and* p *are coprime in case* $p \neq 0$. *If* z *is a complex primitive* k*-th root of unity and* χ *is either a classical character or a* p*-Brauer character of* G, *then for every integer* l *the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d) z^{-dl}\} \qquad (2)$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Finally, we shall use the well-known bound for orders of torsion units.

Proposition 4 (see [23])**.** *The order of a torsion element* $u \in V(\mathbb{Z}G)$ *is a divisor of the exponent of* G.

3. Proof of Theorem 1. Throughout this section we denote the group Ru by G. The character table of G, as well as the p-Brauer character tables, which will be denoted by $\mathfrak{BCT}(p)$ where $p \in \{2, 3, 5, 7, 13, 29\}$, can be found using the computational algebra system GAP [24], which derives these data from [25, 26]. For the characters and conjugacy classes we will use throughout the paper the same notation, indexation inclusive, as used in the GAP Character Table Library.

Since the group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 20, 24, 26 and 29, first of all we investigate units of some these orders (except the units of orders 4, 6, 8, 10, 12, 14, 15, 16, 20, 24 and 26). After this, by Proposition 4, the order of each torsion unit divides the exponent of G, so to prove the Kimmerle's conjecture, it remains to consider units of orders 21, 35, 39, 58, 65, 87, 91, 145, 203 and 377. We will prove that no units of all these orders do appear in $V(\mathbb{Z}G)$.

Now we consider each case separately.

Let u *be an involution*. By (1) and Proposition 2 we have that $\nu_{2a} + \nu_{2b} = 1$. Put $t_1 = 3\nu_{2a} - 7\nu_{2b}$ and $t_2 = 11\nu_{2a} - 7\nu_{2b}$. Applying Proposition 3 we get the following system

$$\mu_1(u, \chi_2, *) = \frac{1}{2}(2t_1 + 378) \geq 0, \qquad \mu_0(u, \chi_2, *) = \frac{1}{2}(-2t_1 + 378) \geq 0,$$
$$\mu_0(u, \chi_4, *) = \frac{1}{2}(2t_2 + 406) \geq 0, \qquad \mu_1(u, \chi_4, *) = \frac{1}{2}(-2t_2 + 406) \geq 0.$$

From these restrictions and the requirement that all $\mu_i(u,\chi_j,*)$ must be non-negative integers we get 22 pairs (ν_{2a},ν_{2b}) listed in part (iii) of the Theorem 1.

Note that using our implementation of the Luthar – Passi method, which we intended to make available in the GAP package LAGUNA [27], we computed inequalities from Proposition 3 for every irreducible character from ordinary and Brauer character tables, and for every $0 \le l \le |u|-1$, but the only inequalities that really matter are those ones listed above. The same remark applies for all other orders of torsion units considered in the paper.

Let u be a unit of order either 3, 7 *or* 13. Using Proposition 2 we obtain that all partial augmentations except one are zero. Thus by Proposition 1 part (ii) of the Theorem 1 is proved.

Let u be a unit of order 5. By (1) and Proposition 2 we get $\nu_{5a}+\nu_{5b}=1$. Put $t_1 = 6\nu_{5a}+\nu_{5b}$ and $t_2 = 3\nu_{5a}-2\nu_{5b}$. By (2) we obtain the system of inequalities

$$\mu_0(u,\chi_4,*) = \frac{1}{5}(4t_1+406) \ge 0, \qquad \mu_1(u,\chi_4,*) = \frac{1}{5}(-t_1+406) \ge 0,$$
$$\mu_0(u,\chi_2,2) = \frac{1}{5}(4t_2+28) \ge 0, \qquad \mu_1(u,\chi_2,2) = \frac{1}{5}(-t_2+28) \ge 0.$$

Again, using the condition for $\mu_i(u,\chi_j,p)$ to be non-negative integers, we obtain eight pairs (ν_{5a},ν_{5b}) listed in part (iv) of the Theorem 1.

Let u be a unit of order 29. By (1) and Proposition 2 we have that $\nu_{29a}+\nu_{29b}=1$. Put $t_1 = 15\nu_{29a}-14\nu_{29b}$. Then using (2) we obtain the system of inequalities

$$\mu_1(u,\chi_6,2) = \frac{1}{29}(t_1+8192) \ge 0, \qquad \mu_2(u,\chi_7,5) = \frac{1}{29}(-t_1+2219) \ge 0,$$
$$\mu_1(u,\chi_2,5) = \frac{1}{29}(12\nu_{29a}-17\nu_{29b}+133) \ge 0,$$
$$\mu_2(u,\chi_2,5) = \frac{1}{29}(-17\nu_{29a}+12\nu_{29b}+133) \ge 0.$$

Now applying the condition for $\mu_i(u,\chi_j,p)$ to be non-negative integers we obtain ten pairs (ν_{29a},ν_{29b}) listed in part (v) of the Theorem 1.

Now it remains to prove part (i) of the Theorem 1.

Let u be a unit of order 21. By (1) and Proposition 2 we obtain that $\nu_{3a}+\nu_{7a}=1$. By (2) we obtain the system of inequalities

$$\mu_1(u,\chi_4,*) = \frac{1}{21}(\nu_{3a}+405) \ge 0,$$
$$\mu_0(u,\chi_2,2) = \frac{1}{21}(12\nu_{3a}+30) \ge 0,$$
$$\mu_7(u,\chi_2,2) = \frac{1}{21}(-6\nu_{3a}+27) \ge 0,$$

which has no integer solutions such that all $\mu_i(u,\chi_j,p)$ are non-negative integers.

Let u be a unit of order 35. By (1) and Proposition 2 we get $\nu_{5a}+\nu_{7a}+\nu_{7b}=1$. Put $t_1 = \nu_{5a}+\nu_{5b}$. Since $|u^7|=5$, for any character χ of G we need to consider eight cases defined by part (iv) of the Theorem 1. Using (2), in all of these cases we get the same system of inequalities

$$\mu_0(u,\chi_2,*) = \frac{1}{35}(72t_1+390) \ge 0,$$

$$\mu_0(u,\chi_4,2) = \frac{1}{35}(-96t_1+1230) \geq 0,$$

which has no integer solutions such that all $\mu_i(u,\chi_j,p)$ are non-negative integers.

Let u be a unit of order 39. By (1) and Proposition 2 we have that $\nu_{3a}+\nu_{13a}=1$. By (2) we obtain that

$$\mu_0(u,\chi_5,*) = \frac{1}{39}(72\nu_{13a}+819) \geq 0,$$
$$\mu_{13}(u,\chi_5,*) = \frac{1}{39}(-36\nu_{13a}+819) \geq 0,$$
$$\mu_1(u,\chi_2,*) = \frac{1}{39}(\nu_{13a}+377) \geq 0,$$
$$\mu_{13}(u,\chi_2,2) = \frac{1}{39}(-12\nu_{3a}-24\nu_{13a}+51) \geq 0.$$

From the first two inequalities we obtain that $\nu_{13a} \in \{0,13\}$, and now the last two inequalities lead us to a contradiction.

Let u be a unit of order 58. By (1) and Proposition 2 we have that

$$\nu_{2a}+\nu_{2b}+\nu_{29a}+\nu_{29b}=1.$$

Put $t_1 = 6\nu_{2a}-14\nu_{2b}-\nu_{29a}-\nu_{29b}$, $t_2 = 11\nu_{2a}-7\nu_{2b}$ and $t_3 = 64\nu_{2b}+14\nu_{29a}-15\nu_{29b}$. Since $|u^2| = 29$ and $|u^{29}| = 2$, according to parts (iii) and (v) of the Theorem 1 we need to consider 220 cases, which we can group in the following way. First, let

$$\chi(u^{29}) \in \Big\{\chi(2a),\ -5\chi(2a)+6\chi(2b),\ -10\chi(2a)+11\chi(2b),$$
$$-2\chi(2a)+3\chi(2b),\ -8\chi(2a)+9\chi(2b),\ 6\chi(2a)-5\chi(2b),$$
$$3\chi(2a)-2\chi(2b),\ 9\chi(2a)-8\chi(2b),\ 4\chi(2a)-3\chi(2b)\Big\}.$$

Then by (2) we obtain the system of inequalities

$$\mu_0(u,\chi_2,*) = \frac{1}{58}(-28t_1+\alpha) \geq 0,$$
$$\mu_{29}(u,\chi_2,*) = \frac{1}{58}(28t_1+\beta) \geq 0,$$
$$\mu_1(u,\chi_2,*) = \frac{1}{58}(-t_1+\gamma) \geq 0,$$

where

$$(\alpha,\beta,\gamma) = \begin{cases} (400,412,383), & \text{if } \chi(u^{29}) = \chi(2a); \\ (520,292,263), & \text{if } \chi(u^{29}) = -5\chi(2a)+6\chi(2b); \\ (620,192,163), & \text{if } \chi(u^{29}) = -10\chi(2a)+11\chi(2b); \\ (460,352,323), & \text{if } \chi(u^{29}) = -2\chi(2a)+3\chi(2b); \\ (580,232,203), & \text{if } \chi(u^{29}) = -8\chi(2a)+9\chi(2b); \\ (300,512,483), & \text{if } \chi(u^{29}) = 6\chi(2a)-5\chi(2b); \\ (360,452,423), & \text{if } \chi(u^{29}) = 3\chi(2a)-2\chi(2b); \\ (240,572,543), & \text{if } \chi(u^{29}) = 9\chi(2a)-8\chi(2b); \\ (340,472,443), & \text{if } \chi(u^{29}) = 4\chi(2a)-3\chi(2b), \end{cases}$$

which has no integral solution such that all $\mu_i(u,\chi_j,p)$ are non-negative integers.

In the remaining cases we consider the following system obtained by (2):

$$\mu_0(u,\chi_2,*) = \frac{1}{58}(-28t_1+\alpha_1) \geq 0,$$

$$\mu_{29}(u,\chi_2,*) = \frac{1}{58}(28t_1+\alpha_2) \geq 0,$$

$$\mu_0(u,\chi_4,*) = \frac{1}{58}(56t_2+\alpha_3) \geq 0,$$

$$\mu_{29}(u,\chi_4,*) = \frac{1}{58}(-56t_2+\alpha_4) \geq 0,$$

$$\mu_1(u,\chi_{34},*) = \frac{1}{58}(-t_3+\beta_1) \geq 0,$$

$$\mu_4(u,\chi_{34},*) = \frac{1}{58}(t_3+\beta_2) \geq 0,$$

where the tuple of coefficients $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ depends only of the value of $\chi(u^{29})$:

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \begin{cases} (420,392,392,420), & \text{if } \chi(u^{29}) = \chi(2b); \\ (320,492,572,240), & \text{if } \chi(u^{29}) = 5\chi(2a)-4\chi(2b); \\ (600,212,68,744), & \text{if } \chi(u^{29}) = -9\chi(2a)+10\chi(2b); \\ (540,272,176,636), & \text{if } \chi(u^{29}) = -6\chi(2a)+7\chi(2b); \\ (380,432,464,348), & \text{if } \chi(u^{29}) = 2\chi(2a)-\chi(2b); \\ (260,552,680,132), & \text{if } \chi(u^{29}) = 8\chi(2a)-7\chi(2b); \\ (480,332,284,528), & \text{if } \chi(u^{29}) = -3\chi(2a)+4\chi(2b); \\ (500,312,248,564), & \text{if } \chi(u^{29}) = -4\chi(2a)+5\chi(2b); \\ (200,612,24,24), & \text{if } \chi(u^{29}) = 11\chi(2a)-10\chi(2b); \\ (220,592,752,60), & \text{if } \chi(u^{29}) = 10\chi(2a)-9\chi(2b); \\ (280,532,644,168), & \text{if } \chi(u^{29}) = 7\chi(2a)-6\chi(2b); \\ (440,372,356,456), & \text{if } \chi(u^{29}) = -\chi(2a)+2\chi(2b); \\ (560,252,140,672), & \text{if } \chi(u^{29}) = -7\chi(2a)+8\chi(2b), \end{cases}$$

while the pair (β_1,β_2) depends both on $\chi(u^{29})$ and $\chi(u^2)$:

	$\chi(29a)$	$\chi(29b)$
$\chi(2b)$	$110641, 110513$	$110670, 110542$
$5\chi(2a) - 4\chi(2b)$	$110321, 110833$	$110350, 110862$
$-9\chi(2a) + 10\chi(2b)$	$111217, 109937$	$111246, 109966$
$-6\chi(2a) + 7\chi(2b)$	$111025, 110129$	$111054, 110158$
$2\chi(2a) - \chi(2b)$	$110513, 110641$	$110542, 110670$
$8\chi(2a) - 7\chi(2b)$	$110129, 111025$	$110158, 111054$
$-3\chi(2a) + 4\chi(2b)$	$110833, 110321$	$110862, 110350$
$-4\chi(2a) + 5\chi(2b)$	$110897, 110257$	$110926, 110286$
$11\chi(2a) - 10\chi(2b)$	$109937, 111217$	$109966, 111246$
$10\chi(2a) - 9\chi(2b)$	$110001, 111153$	$110030, 111182$
$7\chi(2a) - 6\chi(2b)$	$110193, 110961$	$110222, 110990$
$-\chi(2a) + 2\chi(2b)$	$110705, 110449$	$110734, 110478$
$-7\chi(2a) + 8\chi(2b)$	$111089, 110065$	$111118, 110094$

	$5\chi(29a) - 4\chi(29b)$	$-2\chi(29a) + 3\chi(29b)$
$\chi(2b)$	$110525, 110397$	$110728, 110600$
$5\chi(2a) - 4\chi(2b)$	$110205, 110717$	$110408, 110920$
$-9\chi(2a) + 10\chi(2b)$	$111101, 109821$	$111304, 110024$
$-6\chi(2a) + 7\chi(2b)$	$110909, 110013$	$111112, 110216$
$2\chi(2a) - \chi(2b)$	$110397, 110525$	$110600, 110728$
$8\chi(2a) - 7\chi(2b)$	$110013, 110909$	$110216, 111112$
$-3\chi(2a) + 4\chi(2b)$	$110717, 110205$	$110920, 110408$
$-4\chi(2a) + 5\chi(2b)$	$110781, 110141$	$110984, 110344$
$11\chi(2a) - 10\chi(2b)$	$109821, 111101$	$110024, 111304$
$10\chi(2a) - 9\chi(2b)$	$109885, 111037$	$110088, 111240$
$7\chi(2a) - 6\chi(2b)$	$110077, 110845$	$110280, 111048$
$-\chi(2a) + 2\chi(2b)$	$110589, 110333$	$110792, 110536$
$-7\chi(2a) + 8\chi(2b)$	$110973, 109949$	$111176, 110152$

	$2\chi(29a)-$ $-\chi(29b)$	$-3\chi(29a)+$ $+4\chi(29b)$	$-4\chi(29a)+$ $+5\chi(29b)$
$\chi(2b)$	$110612, 110484$	$110757, 110629$	$110786, 110658$
$5\chi(2a) - 4\chi(2b)$	$110292, 110804$	$110437, 110949$	$110466, 110978$
$-9\chi(2a) + 10\chi(2b)$	$111188, 109908$	$111333, 110053$	$111362, 110082$
$-6\chi(2a) + 7\chi(2b)$	$110996, 110100$	$111141, 110245$	$111170, 110274$
$2\chi(2a) - \chi(2b)$	$110484, 110612$	$110629, 110757$	$110658, 110786$
$8\chi(2a) - 7\chi(2b)$	$110100, 110996$	$110245, 111141$	$110274, 111170$
$-3\chi(2a) + 4\chi(2b)$	$110804, 110292$	$110949, 110437$	$110978, 110466$
$-4\chi(2a) + 5\chi(2b)$	$110868, 110228$	$111013, 110373$	$111042, 110402$
$11\chi(2a) - 10\chi(2b)$	$109908, 111188$	$110053, 111333$	$110082, 111362$
$10\chi(2a) - 9\chi(2b)$	$109972, 111124$	$110117, 111269$	$110146, 111298$
$7\chi(2a) - 6\chi(2b)$	$110164, 110932$	$110309, 111077$	$110338, 111106$
$-\chi(2a) + 2\chi(2b)$	$110676, 110420$	$110821, 110565$	$110850, 110594$
$-7\chi(2a) + 8\chi(2b)$	$111060, 110036$	$111205, 110181$	$111234, 110210$

	$3\chi(29a)-$ $-2\chi(29b)$	$-\chi(29a)+$ $+2\chi(29b)$	$4\chi(29a)-$ $-3\chi(29b)$
$\chi(2b)$	$110583, 110455$	$110699, 110571$	$110554, 110426$
$5\chi(2a) - 4\chi(2b)$	$110263, 110775$	$110379, 110891$	$110234, 110746$
$-9\chi(2a) + 10\chi(2b)$	$111159, 109879$	$111275, 109995$	$111130, 109850$
$-6\chi(2a) + 7\chi(2b)$	$110967, 110071$	$111083, 110187$	$110938, 110042$
$2\chi(2a) - \chi(2b)$	$110455, 110583$	$110571, 110699$	$110426, 110554$
$8\chi(2a) - 7\chi(2b)$	$110071, 110967$	$110187, 11108$	$110042, 110938$
$-3\chi(2a) + 4\chi(2b)$	$110775, 110263$	$110891, 110379$	$110746, 110234$
$-4\chi(2a) + 5\chi(2b)$	$110839, 110199$	$110955, 110315$	$110810, 110170$
$11\chi(2a) - 10\chi(2b)$	$109879, 111159$	$109995, 111275$	$109850, 111130$
$10\chi(2a) - 9\chi(2b)$	$109943, 111095$	$110059, 111211$	$109914, 111066$
$7\chi(2a) - 6\chi(2b)$	$110135, 110903$	$110251, 111019$	$110106, 110874$
$-\chi(2a) + 2\chi(2b)$	$110647, 110391$	$110763, 110507$	$110618, 110362$
$-7\chi(2a) + 8\chi(2b)$	$111031, 110007$	$111147, 110123$	$111002, 109978$

Additionally, when $\chi(u^{29}) \in \{\chi(2b), 7\chi(2a)-6\chi(2b), -7\chi(2a)+8\chi(2b)\}$, we need to consider one more inequality

$$\mu_1(u, \chi_2, *) = \frac{1}{58}(-6\nu_{2a} + 14\nu_{2b} + \nu_{29a} + \nu_{29b} + \gamma) \geq 0,$$

where

$$\gamma = \begin{cases} 363, & \text{if } \chi(u^{29}) = \chi(2b); \\ 503, & \text{if } \chi(u^{29}) = 7\chi(2a) - 6\chi(2b); \\ 223, & \text{if } \chi(u^{29}) = -7\chi(2a) + 8\chi(2b). \end{cases}$$

All systems of inequalities, constructed as described above, have no integer solutions such that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

Let u be a unit of order 65. By (1) and Proposition 2 we have that

$$\nu_{5a} + \nu_{5b} + \nu_{13a} = 1.$$

Since $|u^{13}| = 5$, we need to consider eight cases listed in part (iv) of the Theorem 1. Put $t_1 = 3\nu_{5a} + 3\nu_{5b} + \nu_{13a}$ and $t_2 = 6\nu_{5a} + \nu_{5b} + 3\nu_{13a}$. Then using (2) we obtain

$$\mu_0(u, \chi_2, *) = \frac{1}{65}(48t_1 + 402) \geq 0,$$

$$\mu_{13}(u, \chi_2, *) = \frac{1}{65}(-12t_1 + 387) \geq 0,$$

$$\mu_0(u, \chi_4, *) = \frac{1}{65}(48t_2 + \alpha) \geq 0,$$

$$\mu_{13}(u, \chi_4, *) = \frac{1}{65}(-12t_2 + \beta) \geq 0,$$

where

$$(\alpha, \beta) = \begin{cases} (466, 436), & \text{if } \chi(u^{13}) = \chi(5a); \\ (446, 441), & \text{if } \chi(u^{13}) = \chi(5b); \\ (546, 416), & \text{if } \chi(u^{13}) = 5\chi(5a) - 4\chi(5b); \\ (486, 431), & \text{if } \chi(u^{13}) = 2\chi(5a) - \chi(5b); \\ (566, 411), & \text{if } \chi(u^{13}) = 6\chi(5a) - 5\chi(5b); \\ (506, 426), & \text{if } \chi(u^{13}) = 3\chi(5a) - 2\chi(5b); \\ (426, 446), & \text{if } \chi(u^{13}) = -\chi(5a) + 2\chi(5b); \\ (526, 421), & \text{if } \chi(u^{13}) = 4\chi(5a) - 3\chi(5b). \end{cases}$$

In all cases we have no solutions such that all $\mu_i(u, \chi_i, p)$ are non-negative integers.

Let u be a unit of order 87. By (1) and Proposition 2 we have that

$$\nu_{3a} + \nu_{29a} + \nu_{29b} = 1.$$

Since $|u^3| = 29$, according to part (v) of the Theorem 1 we need to consider ten cases. Put $t_1 = \nu_{29a} + \nu_{29b}$. In all of these cases by (2) we get the system

$$\mu_0(u, \chi_2, *) = \frac{1}{87}(56t_1 + 406) \geq 0,$$

$$\mu_{29}(u, \chi_2, *) = \frac{1}{87}(-28t_1 + 406) \geq 0,$$

that lead us to a contradiction.

Let u be a unit of order 91. By (1) and Proposition 2 we get $\nu_{7a} + \nu_{13a} = 1$. Now using (2) we obtain non-compatible inequalities

$$\mu_0(u,\chi_2,2) = \frac{1}{91}(144\nu_{13a} + 52) \geq 0,$$

$$\mu_7(u,\chi_2,2) = \frac{1}{91}(-12\nu_{13a} + 26) \geq 0.$$

Let u be a unit of order 145. By (1) and Proposition 2 we have that

$$\nu_{5a} + \nu_{5b} + \nu_{29a} + \nu_{29b} = 1.$$

Put $t_1 = 3\nu_{5a} + 3\nu_{5b} + \nu_{29a} + \nu_{29b}$. Since $|u^{29}| = 5$ and $|u^5| = 29$, for any character χ of G we need to consider 80 cases defined by parts (iv) and (v) of the Theorem 1. Luckily, in every case by (2) we obtain the same pair of incompatible inequalities

$$\mu_0(u,\chi_2,*) = \frac{1}{145}(112t_1 + 418) \geq 0,$$

$$\mu_{29}(u,\chi_2,*) = \frac{1}{145}(-28t_1 + 403) \geq 0.$$

Let u be a unit of order 203. By (1) and Proposition 2 we have that

$$\nu_{7a} + \nu_{29a} + \nu_{29b} = 1.$$

Since $|u^7| = 29$, according to part (v) of the Theorem 1 we need to consider ten cases. Put $t_1 = \nu_{29a} + \nu_{29b}$, and then using (2) in each case we obtain a non-compatible system of inequalities

$$\mu_{29}(u,\chi_2,2) = \frac{1}{203}(28t_1) \geq 0,$$

$$\mu_0(u,\chi_2,2) = \frac{1}{203}(-168t_1) \geq 0,$$

$$\mu_1(u,\chi_2,*) = \frac{1}{203}(t_1 + 377) \geq 0.$$

Let u be a unit of order 377. By (1) and Proposition 2 we have that

$$\nu_{13a} + \nu_{29a} + \nu_{29b} = 1.$$

Since $|u^{13}| = 29$, we need to consider ten cases defined by part (v) of the Theorem 1. In each case by (2) we obtain the following system of inequalities

$$\mu_0(u,\chi_4,*) = \frac{1}{377}(1008\nu_{13a} + 442) \geq 0,$$

$$\mu_{29}(u,\chi_4,*) = \frac{1}{377}(-84\nu_{13a} + 403) \geq 0.$$

which have no solution such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

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