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ON AN INVARIANT ON ISOMETRIC IMMERSIONS INTO SPACES OF CONSTANT SECTIONAL CURVATURE*

ПРО ІНВАРІАНТ НА ІЗОМЕТРИЧНИХ ЗАНУРЕННЯХ У ПРОСТОРИ СТАЛОЇ СЕКЦІЙНОЇ КРИВИНИ

In the present paper we give an invariant on isometric immersions into spaces of constant sectional curvature. This invariant is a direct consequence of the Gauss equation and the Codazzi equation of isometric immersions. We will apply this invariant on some examples. Further we will apply it to codimension 1 local isometric immersions of 2-step nilpotent Lie groups with arbitrary left invariant Riemannian metric into spaces of constant non-positive sectional curvature. We will also consider the more general class: Three dimensional Lie groups G with non-trivial center and with arbitrary left-invariant metric. We show that when the metric of G is not symmetric, there are no local isometric immersions of G into Q_c^4.

Наведено інваріант на ізометричних зануреннях у простори сталої секційної кривини. Цей інваріант є наслідком рівняння Гаусса та рівняння Кодацці для ізометричного занурення. Цей інваріант використано у декількох прикладах. Його застосовано до локальних ізометричних занурень ковимірності 1 2-крокових нільпотентних груп Лі з довільною інваріантною зліва рімановою метрикою у простір сталої недодатної секційної кривини. Розглянуто також більш загальний клас, а саме тривимірні групи Лі G із нетривіальним центром та довільною інваріантною зліва метрикою. Показано, що у випадку, коли метрика G несиметрична, не існує локальних ізометричних занурень G у Q_c^4.

1. Introduction. A special case of a result, due to Otsuki [1] is as follows. If the sectional curvature of a Riemannian manifold is strictly negative, then any local isometric immersion of the Riemannian manifold into a Euclidean space is of codimension greater or equal to one less than the dimension of the manifold. The sectional curvature of a 2-step nilpotent group is not strictly negative. By studying the Gauss equation for the curvature tensor of 2-step nilpotent groups with arbitrary left invariant metric, it has been shown [2] that there exists no isometric immersions of open subsets of the 2-step nilpotent groups into Euclidean space.

Recently Masal'tsev [3] proved that there are no isometric immersions of any region of the three dimensional Heisenberg group into any space form of constant sectional curvature. In the present article we shall prove that no left invariant nonsymmetric metrics of the three dimensional Lie groups of Bianchi type II and III are not immersable into a four dimensional space-forms of constant sectional curvature.

2. A new invariant. Let $*$ denote the Hodge star operator, and let $R\colon \Lambda^2 TM \to \Lambda^2 TM$ denote the curvature operator of a Riemannian manifold M. That is $\langle R(X \wedge Y), Z \wedge W\rangle = \langle R(X, Y)W, Z\rangle$.

Theorem 1. *Let M be a three dimensional Riemannian manifold and let p be an arbitrary chosen point in M. Let Ξ is an endomorphism of T_pM which permuting two of three vectors and fixes the third vector in some chosen basis $\{e_1, e_2, e_3\}$ of T_pM. Let $C_\Xi\colon T_pM \to \Lambda^2 T_pM$ be the map defined by $C_\Xi\colon X_p \mapsto (\nabla_{\Xi(X_p)}R)(\Xi(X_p)\wedge X_p)$. Define the number $f_{M,p}$ by*

$$f_{M,p} = \sum_{i=1}^{3} \langle * \circ (R - cI\wedge I)^t_{cf} \circ C_\Xi(e_i), e_i\rangle,$$

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where $(R - cI \wedge I)^t_{cf} : \Lambda^2(T_pM) \to \Lambda^2(T_pM)$ *denotes the transposed cofactor operator of* $R - cI \wedge I$ *with respect to the basis* $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$. *If a neighborhood of* p *in* M *is isometric immersable into* Q^4_c, *we have* $f_{M,p} = 0$.

***Remark* 1.** The invariant $f_{M,p}$ in Theorem 1 was found by a method given in my thesis [4]. The method used in [4] generalized a method by Agaoka [5] which uses $GL(n)$-equivariance of the Gauss equation. A paper that extends the method from my thesis is in preparation [6].

***Corollary* 1.** *Let* M *be a Riemannian space of dimension* 3 *with metric* $\langle , \rangle$ *and let* p *be an arbitrary chosen point in* M. *Let* X_1, X_2 *and* X_3 *be arbitrary vectors in* T_pM.

Define the function $f_{M,p}$ *by*

$$f_{M,p}(X_1, X_2, X_3) = R^2_{1213}C_{21223} - R_{1313}R_{1212}C_{21223} - R_{1223}R_{1213}C_{11223} +$$
$$+ R_{1323}R_{1212}C_{11223} - R_{1213}R_{1223}C_{21213} + R_{1323}R_{1212}C_{21213} +$$
$$+ R^2_{1223}C_{11213} - R_{2323}R_{1212}C_{11213} + R_{1313}R_{1223}C_{21212} -$$
$$- R_{1323}R_{1213}C_{21212} - R_{1323}R_{1223}C_{11212} + R_{2323}R_{1213}C_{11212},$$

where

$$C_{ijklm} = \langle (\nabla_{X_i} R)(X_j, X_k)X_l, X_m \rangle$$

and

$$R_{ijkl} = \langle R(X_i, X_j)X_k, X_l \rangle - c\langle X_i, X_l\rangle\langle X_j, X_k\rangle + c\langle X_i, X_k\rangle\langle X_j, X_l\rangle$$

for $i, j, k, l, m = 1, 2, 3$. *If an open neighborhood of* p *in* M *is isometric immersable into a* 4*-dimensional space* Q^4_c *of arbitrary constant sectional curvature* c, *we have* $f_{M,p}(X_1, X_2, X_3) = 0$ *for all* X_1, X_2, *and* $X_3 \in T_pM$.

***Remark* 2.** In the corollary, we have used a Ξ which interchanges X_1 and X_2.

Theorem 2. *Let* G *be an* 3*-dimensional* 2*-step nilpotent group* G *equipped with an arbitrary left invariant metric. If there is a set* X, Y, Z *of* $O.N.$ *left invariant vector fields such that* $[X, Y] = Z$ *and with* $|\mathrm{ad}(X)^*Z| \neq |\mathrm{ad}(Y)^*Z|$ *or* $c \neq \dfrac{\mathrm{ad}(X)^*Z}{4|Z|^2}$, *then there exists no isometric immersions of an open set of* G *into a space* Q^4_c *of sectional curvature* c.

We will prove this theorem in Subsection 4.2.

3. On local isometric immersions. Let M denote a Riemannian manifold and let p be a fixed point in M. Given a local isometric immersion of M into Q^4_c. Let α be its second fundamental form and let β be the covariant derivative of α. Recall that by the Codazzi equation, β is symmetric. Define L and B by $\langle L(X), Y\rangle = \langle \alpha(X, Y), \xi\rangle$ and $\langle B(X, Y), Z\rangle = \langle \beta(X, Y, Z), \xi\rangle$, where $X, Y, Z \in T_pG$ are tangent vectors at e and $\xi \in N_pG$ is a normal vector of the immersion at p.

***Proof of Theorem* 1.** The Gauss equation for isometric immersions into spaces of constant sectional curvature c is $R(X \wedge Y) = -L(X) \wedge L(Y) + cX \wedge Y$ (see, e.g., [7].) The cofactor version of the Gauss equation is $(R - cI \wedge I)^t_{cf} = -L^t_{cf} \wedge L^t_{cf}$. The prolonged Gauss – Codazzi equation is $(\nabla_T R) = -L \wedge B(T, \bullet) - B(T, \bullet) \wedge L$ (see [8] for the case of Euclidean ambient space and Corollary 2 in the present article for the case when the ambient space has constant sectional curvature). Hence, $C_\Xi(X) = -L(\Xi(X)) \wedge B(\Xi(X), X) - B(\Xi(X), \Xi(X)) \wedge L(X)$. Therefore,

$$(R - cI \wedge I)^t_{cf} \circ C_\Xi(X) = \det(L)\Xi(X) \wedge L^t_{cf} \circ B(\Xi(X), X) +$$

$$+ \det(L) L^t_{cf} \circ B(\Xi(X), \Xi(X)) \wedge X.$$

Now, since $\langle *(X \wedge Y), Z\rangle = \det(X|Y|Z)$, we have

$$\langle * \circ (R - cI \wedge I)^t_{cf} \circ C_\Xi(X), X\rangle = \det(L)\det(\Xi(X)| L^t_{cf} \circ B(\Xi(X), X)| X) +$$

$$+ \det(L)\det(L^t_{cf} \circ B(\Xi(X), \Xi(X))| X | X) =$$

$$= \det(L)\det(L^t_{cf})\det(L(\Xi(X))| B(\Xi(X), X)| L(X)) - 0 =$$

$$= -\det(L)^3 \det(L(\Xi(X)| L(X)| B(\Xi(X), X))) =$$

$$= \det(L)^3 \langle - * \circ L \wedge L(\Xi(X) \wedge X), B(\Xi(X), X)\rangle =$$

$$= \det(L)^3 \langle * \circ R(\Xi(X) \wedge X) - c\Xi(X) \wedge X), B(\Xi(X), X)\rangle.$$

We can assume that Ξ permutes e_1 and e_2. Therefore,

$$f_{M,p} = \det(L)^3 \langle *(R(e_2 \wedge e_1) - ce_2 \wedge e_1), B(e_2, e_1)\rangle +$$

$$+ \det(L)^3 \langle *(R(e_1 \wedge e_2) - ce_1 \wedge e_2), B(e_1, e_2)\rangle = 0.$$

The theorem is proved.

4. On 2-step nilpotent groups. ***4.1. On the Levi-Civita connection of 2-step nilpotent groups.*** We recall some facts about 2-step nilpotent groups. Let $\mathfrak{g}$ be a Lie algebra. Define $\mathfrak{g}_i$, $i \geq 0$, recursively by $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_n = [\mathfrak{g}_{n-1}, \mathfrak{g}]$ for $n \in \mathbb{N}$.

Definition. *A Lie algebra is called nilpotent if* $\mathfrak{g}_n = \{0\}$ *for some integer* n. *If* $\mathfrak{g}_k = \{0\}$ *and* $\mathfrak{g}_{k-1} \neq \{0\}$ *then* $\mathfrak{g}$ *is called* k*-step nilpotent. A Lie group* G *is called* k*-step nilpotent if its Lie algebra is* k*-step nilpotent.*

Let G be a Lie group equipped with an arbitrary left invariant inner product $\langle,\rangle$. If X and Y are left invariant vector fields on G then $\langle X, Y\rangle$ is constant. Therefore,

$$\nabla_X Y = \frac{1}{2}[X, Y] - \frac{1}{2}\operatorname{ad}(X)^* Y - \frac{1}{2}\operatorname{ad}(Y)^* X, \tag{1}$$

where $\operatorname{ad}(X)^*$ denotes the adjoint map of $\operatorname{ad}(X)$.

Let G be a 2-step nilpotent Lie group. Let X and Y be orthonormal left invariant vector fields perpendicular to $\mathfrak{g}_1$ such that $[X, Y] \neq 0$. Let $Z = [X, Y]$. We easily compute $\nabla_X Y = -\nabla_Y X = \frac{1}{2} Z$, $\nabla_X Z = \nabla_Z X = -\frac{1}{2}\operatorname{ad}(X)^* Y \perp \mathfrak{g}_1$, $\nabla_Y Z = \nabla_Z Y = -\frac{1}{2}\operatorname{ad}(Y)^* Z \perp \mathfrak{g}_1$, and $\nabla_X X = \nabla_Y Y = \nabla_Z Z = 0$. Since Z is independent of rotations of X and Y we may replace X and Y with the orthonormal eigen vectors of the symmetric bilinear form

$$(U, V) \mapsto \langle \operatorname{ad}(U)^* Z, \operatorname{ad}(V)^* Z \rangle$$

on span $\{X, Y\}$. From [2] we have that

$$\begin{aligned}
\langle R(X,Y)X - \tilde{R}(X,Y)X, Y \rangle &= \frac{3}{4}|Z|^2 + c,\\
\langle R(X,Y)X - \tilde{R}(X,Y)X, Z \rangle &= 0,\\
\langle R(X,Y)Y - \tilde{R}(X,Y)Y, Z \rangle &= 0,\\
\langle R(X,Z)X - \tilde{R}(X,Z)X, Z \rangle &= -\frac{1}{4}\left|\operatorname{ad}(X)^* Z\right|^2 + c|Z|^2,\\
\langle R(X,Z)Y - \tilde{R}(X,Z)Y, Z \rangle &= 0,\\
\langle R(Y,Z)Y - \tilde{R}(Y,Z)Y, Z \rangle &= -\frac{1}{4}\left|\operatorname{ad}(Y)^* Z\right|^2 + c|Z|^2,
\end{aligned} \tag{2}$$

where $\tilde{R}$ denotes the curvature tensor of a space form of constant sectional curvature c. We easily compute the covariant derivatives of the curvature tensor by using the formulas in (2) and the formula (1): The nonzero components are

$$\begin{aligned}
\langle (\nabla_X R)(X,Y)X, Z \rangle &= \frac{1}{2}\left|\operatorname{ad}(X)^* Z\right|^2,\\
\langle (\nabla_Y R)(X,Y)Y, Z \rangle &= \frac{1}{2}\left|\operatorname{ad}(Y)^* Z\right|^2.
\end{aligned} \tag{3}$$

We have only considered formulas involving X, Y, and Z.

***4.2.* Proof of Theorem 2.** Let X, Y be orthonormal left invariant vector fields perpendicular to $\mathfrak{g}_1$, and let $Z = [X, Y]$. The invariant $f_{M,p}$ for arbitrary $p \in G = M$ in Corollary 1 together with the formulas in (1) and (2) gives

$$f_{G,p}(X,Z,Y) = -R_{XZXZ} R_{YZYZ} C_{XXYXZ} = 0,$$

$$f_{G,p}(X,Y,Z) = -R_{XZXZ} R_{XYXY} C_{YXYYZ} - R_{YZYZ} R_{XYXY} C_{XXYXZ} = 0.$$

The last pair of equations yields $\left|\operatorname{ad}(X)^* Z\right| = \left|\operatorname{ad}(Y)^* Z\right|$ and $c = \dfrac{\left|\operatorname{ad}(X)^* Z\right|^2}{4|Z|^2}$.

The theorem is proved.

5. An application of the main theorem on Lie groups of nontrivial center. Let G be a three dimensional Lie group with an arbitrary left invariant metric $\langle , \rangle$ and with a nontrivial center of its Lie algebra. Let $\mathbf{e}_1 \in \mathfrak{g}$ be a left invariant unit vector field on G contained in the center $Z(\mathfrak{g})$ of $\mathfrak{g}$. From (1) one has $\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0$. One can extend $\{\mathbf{e}_1\}$ to an orthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of left invariant vector fields on G. Let Γ_{ij}^k be the k-th coefficient of $\nabla_{\mathbf{e}_i}\mathbf{e}_j$. From the compatibility conditions of ∇, we have $\Gamma_{ij}^k = \langle \nabla_{\mathbf{e}_i}\mathbf{e}_j, \mathbf{e}_k \rangle = -\langle \nabla_{\mathbf{e}_i}\mathbf{e}_k, \mathbf{e}_j \rangle = -\Gamma_{ik}^j$. The identities $\Gamma_{1j}^k = \Gamma_{j1}^k$ and $\Gamma_{11}^k = \Gamma_{22}^1 = \Gamma_{33}^1 = 0$ follows from the equation (1) and $\mathbf{e}_1 \in Z(\mathfrak{g})$. An easy calculation gives

$$\begin{aligned}
\langle R(\mathbf{e}_1, \mathbf{e}_2)\mathbf{e}_1 - \tilde{R}(\mathbf{e}_1, \mathbf{e}_2)\mathbf{e}_1, \mathbf{e}_2 \rangle &= c - (\Gamma_{23}^1)^2, \\
\langle R(\mathbf{e}_1, \mathbf{e}_3)\mathbf{e}_1 - \tilde{R}(\mathbf{e}_1, \mathbf{e}_3)\mathbf{e}_1, \mathbf{e}_3 \rangle &= c - (\Gamma_{23}^1)^2, \\
\langle R(\mathbf{e}_2, \mathbf{e}_3)\mathbf{e}_2 - \tilde{R}(\mathbf{e}_2, \mathbf{e}_3)\mathbf{e}_2, \mathbf{e}_3 \rangle &= c - (\Gamma_{23}^1)^2 + \left|[\mathbf{e}_2, \mathbf{e}_3]\right|^2
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
\langle (\nabla_{\mathbf{e}_2} R)(\mathbf{e}_1, \mathbf{e}_2)\mathbf{e}_2, \mathbf{e}_3 \rangle &= -\Gamma_{23}^1 \left|[\mathbf{e}_2, \mathbf{e}_3]\right|^2, \\
\langle (\nabla_{\mathbf{e}_3} R)(\mathbf{e}_1, \mathbf{e}_3)\mathbf{e}_2, \mathbf{e}_3 \rangle &= -\Gamma_{23}^1 \left|[\mathbf{e}_2, \mathbf{e}_3]\right|^2,
\end{aligned} \tag{5}$$

where we only have displayed the nonzero formulas up to the symmetries of R, $\tilde{R}$, and ∇R. We now calculate the invariant $f_{G,p}$ in Corollary 1:

$$f_{G,p}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \Gamma_{23}^1 (c - (\Gamma_{23}^1)^2)^2 \left|[\mathbf{e}_2, \mathbf{e}_3]\right|^2.$$

Proposition 1. *Let G be a three dimensional Lie group with a nontrivial center of its Lie algebra $\mathfrak{g}$ and with a left invariant metric $\langle , \rangle$ such that the center is not perpendicular to the derived algebra $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. If*

$$c \neq (\Gamma_{23}^1)^2,$$

then there exists no isometric immersions of any region of G into Q_c^4. In particular $\Gamma_{23}^1 \neq 0$, so there exists no isometric immersions of any region of G into $\mathbb{R}^4$.

***Remark* 3.** For $G = Nil^3$ one has $\mathfrak{g}_1 = Z(\mathfrak{g})$, so the theorem applies for all left invariant geometries on Nil^3.

***Proof of Proposition* 1.** Since $\mathfrak{g}_1 = \operatorname{span}\{[\mathbf{e}_2, \mathbf{e}_3]\}$ and $Z(\mathfrak{g}) \not\perp \mathfrak{g}_1$, we have $\Gamma_{23}^1 \neq 0$ and $|[\mathbf{e}_2, \mathbf{e}_3]| \neq 0$. Thus, $f_{G,p}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \neq 0$, and by Theorem 1 there are no isometric immersion of any region of G into Q_c^4.

This is how long we come by using our invariant $f_{G,p}$. Let $c = (\Gamma_{23}^1)^2$. By the same methods as in [3], we show that the following lemma.

Lemma. *Let G be as in Proposition* 1. *The second fundamental form of the isometric immersion of G into Q_c^4 is on the form*

$$L = \left|[\mathbf{e}_2, \mathbf{e}_3]\right| \begin{vmatrix} 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{vmatrix},$$

where θ is a smooth function on G.

Proof. The Gauss equations yield $L_{11} = L_{12} = L_{13} = 0$. The Codazzi equation, $0 = \nabla_{\mathbf{e}_2} L(\mathbf{e}_3, \mathbf{e}_1) - \nabla_{\mathbf{e}_3} L(\mathbf{e}_2, \mathbf{e}_1)$, reduces to $-\Gamma_{23}^1 L_{22} = \Gamma_{23}^1 L_{33}$. Therefore, $L_{22} = -L_{33}$. The third equation in (4) now gives the result.

The lemma is proved.

The covariant derivative of L is

$$\nabla_{\mathbf{e}_1} L = \left(\mathbf{e}_1(\theta) - 2\Gamma_{13}^2\right) \begin{vmatrix} 0 & 0 & 0 \\ 0 & -L_{23} & L_{22} \\ 0 & L_{22} & L_{23} \end{vmatrix},$$

$$\nabla_{\mathbf{e}_2} L = \begin{vmatrix} 0 & \Gamma_{13}^2 L_{23} & -\Gamma_{13}^2 L_{22} \\ \Gamma_{13}^2 L_{23} & -L_{23}\left(\mathbf{e}_2(\theta) + 2\Gamma_{22}^3\right) & L_{22}\left(\mathbf{e}_2(\theta) + 2\Gamma_{22}^3\right) \\ -\Gamma_{13}^2 L_{22} & L_{22}\left(\mathbf{e}_2(\theta) + 2\Gamma_{22}^3\right) & L_{23}\left(\mathbf{e}_2(\theta) + 2\Gamma_{22}^3\right) \end{vmatrix},$$

$$\nabla_{\mathbf{e}_3} L = \begin{vmatrix} 0 & -\Gamma_{13}^2 L_{22} & -\Gamma_{13}^2 L_{23} \\ -\Gamma_{13}^2 L_{22} & -L_{23}\left(\mathbf{e}_3(\theta) - 2\Gamma_{33}^2\right) & L_{22}\left(\mathbf{e}_3(\theta) - 2\Gamma_{33}^2\right) \\ -\Gamma_{13}^2 L_{23} & L_{22}\left(\mathbf{e}_3(\theta) - 2\Gamma_{33}^2\right) & L_{23}\left(\mathbf{e}_3(\theta) - 2\Gamma_{33}^2\right) \end{vmatrix}.$$

The nontrivial Codazzi equations are:

$$0 = (\nabla_{\mathbf{e}_1} L)_{22} - (\nabla_{\mathbf{e}_2} L)_{12} = -\left(\mathbf{e}_1(\theta) - \Gamma_{13}^2\right) L_{23},$$

$$0 = (\nabla_{\mathbf{e}_1} L)_{23} - (\nabla_{\mathbf{e}_2} L)_{13} = \left(\mathbf{e}_1(\theta) - \Gamma_{13}^2\right) L_{22},$$

$$0 = (\nabla_{\mathbf{e}_1} L)_{32} - (\nabla_{\mathbf{e}_3} L)_{12} = \left(\mathbf{e}_1(\theta) - \Gamma_{13}^2\right) L_{22},$$

$$0 = (\nabla_{\mathbf{e}_1} L)_{33} - (\nabla_{\mathbf{e}_3} L)_{13} = \left(\mathbf{e}_1(\theta) - \Gamma_{13}^2\right) L_{23},$$

$$0 = (\nabla_{\mathbf{e}_2} L)_{32} - (\nabla_{\mathbf{e}_3} L)_{22} = L_{22}\left(\mathbf{e}_2(\theta) + 2\Gamma_{22}^3\right) + L_{23}\left(\mathbf{e}_3(\theta) - 2\Gamma_{33}^2\right),$$

$$0 = (\nabla_{\mathbf{e}_2} L)_{33} - (\nabla_{\mathbf{e}_3} L)_{23} = L_{23}\left(\mathbf{e}_2(\theta) + 2\Gamma_{22}^3\right) - L_{22}\left(\mathbf{e}_3(\theta) - 2\Gamma_{33}^2\right).$$

Therefore, the Codazzi equations are equivalent with

$$d\theta(\mathbf{e}_1) = \Gamma_{13}^2,$$
$$d\theta(\mathbf{e}_2) = -2\Gamma_{22}^3, \tag{6}$$
$$d\theta(\mathbf{e}_3) = 2\Gamma_{33}^2.$$

Let $\omega_1, \omega_2, \omega_3$ be the dual forms of the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Hence, the equations in (6) can be written on the compact form

$$d\theta = \Gamma_{13}^2\omega_1 - 2\Gamma_{22}^3\omega_2 + 2\Gamma_{33}^2\omega_3.$$

The integrability condition for is

$$0 = \Gamma_{13}^2 d\omega_1 - 2\Gamma_{22}^3 d\omega_2 + 2\Gamma_{33}^2 d\omega_3. \tag{7}$$

We calculate $d\omega_1 = -2\Gamma_{13}^2\mathbf{e}_2 \wedge \mathbf{e}_3$, $d\omega_2 = \Gamma_{22}^3\mathbf{e}_2 \wedge \mathbf{e}_3$, and $d\omega_2 = -\Gamma_{33}^2\mathbf{e}_2 \wedge \mathbf{e}_3$. By substituting these for $d\omega_k$, $k = 1, 2, 3$, in (7) we get

$$0 = -2(\Gamma_{23}^1)^2 - 2(\Gamma_{22}^3)^2 - 2(\Gamma_{33}^2)^2. \tag{8}$$

So, condition (8) is necessary for the solutions of the Codazzi equation. This condition is impossible since we have assumed that $\Gamma_{23}^1 \neq 0$. Therefore:

Theorem 3. *Let G be a three dimensional Lie group with a nontrivial center of its Lie algebra $\mathfrak{g}$. Let G haves a left invariant metric $\langle , \rangle$. If the center of $\mathfrak{g}$ is not perpendicular to the derived algebra $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, then there are no isometric immersions of any region of G into Q_c^4.*

***Remark* 4.** If $\mathfrak{g}_1 \perp Z(\mathfrak{g})$, then $\Gamma_{23}^1 = 0$ and hence, from (5), G is local symmetric. This case is not considered in this article.

6. The Gauss – Codazzi equation. Let M be an n-dimensional manifold, and let $\tilde{N}$ be an m-dimensional manifold. In this appendix we will state first prolongation of the Gauss and Codazzi equations. The special case for isometric immersions into Euclidean space is proved by Kaneda [8].

Let $\langle \bullet, \bullet \rangle_M$ and $\langle \bullet, \bullet \rangle$ be Riemannian metrics on the manifolds M and $\tilde{N}$ respectively. Let f be an isometric immersion between the Riemannian manifolds M and $\tilde{N}$ with Levi-Civita connections ∇ and $\tilde{\nabla}$ respectively.

Proposition 2 (First prolonged Gauss – Codazzi equation). *Let f be an isometric immersion of M into $\tilde{N}$. The following equation is satisfied*:

$$\begin{aligned}
\langle (\tilde{\nabla}_V \tilde{R})(X,Y)Z, W \rangle = \langle (\nabla_V R)(X,Y)Z, W \rangle - \\
- \langle (\nabla_V^\perp \alpha)(X,W), \alpha(Y,Z) \rangle - \langle (\nabla_V^\perp \alpha)(Y,Z), \alpha(X,W) \rangle + \\
+ \langle (\nabla_V^\perp \alpha)(X,Z), \alpha(Y,W) \rangle + \langle (\nabla_V^\perp \alpha)(Y,W), \alpha(X,Z) \rangle - \\
- \langle \tilde{R}(X,Y)Z, \alpha(W,V) \rangle + \langle \tilde{R}(X,Y)W, \alpha(Z,V) \rangle + \\
+ \langle \tilde{R}(Z,W)Y, \alpha(X,V) \rangle - \langle \tilde{R}(Z,W)X, \alpha(Y,V) \rangle,
\end{aligned}$$

where α is the second fundamental form of the immersion.

Proof. The proof is a result of a straight forward calculation.

Let A be the shape operator (see, e.g., [9]).

***Corollary* 2.** Let $\{\xi_1, \ldots, \xi_{m-n}\}$ *be a local orthonormal frame of unit normal vectors, and assume that* $\tilde{N}$ *is of constant sectional curvature. We then have*

$$0 = (\nabla_V R)(X, Y) + \sum_i (A_{\xi_i} X) \wedge (\nabla_V A)_{\xi_i} Y - \sum_i (A_{\xi_i} Y) \wedge (\nabla_V A)_{\xi_i} X .$$

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