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WEAK α-SKEW ARMENDARIZ IDEAL

СЛАБКІ α-КОСІ ІДЕАЛИ АРМЕНДАРІЗА

We introduce the concept of weak α-skew Armendariz ideals and investigate their properties. Moreover, we prove that I is a weak α-skew Armendariz ideal if and only if $I[x]$ is a weak α-skew Armendariz ideal. As a consequence, we show that R is a weak α-skew Armendariz ring if and only if $R[x]$ is a weak α-skew Armendariz ring.

Введено поняття слабких α-косих ідеалів Армендаріза та досліджено їх властивості. Крім того, доведено, що I є слабким α-косим ідеалом Армендаріза тоді і тільки тоді, коли $I[x]$ є слабким α-косим ідеалом Армендаріза. Як наслідок, показано, що R є слабким α-косим кільцем Армендаріза тоді і тільки тоді, коли $R[x]$ є слабким α-косим кільцем Армендаріза.

1. Introduction. In [11], Rege and Chhawchharia introduced the notion of an Armendariz ring. They defined a ring R (associative with identity) to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \ldots + a_mx^m$, $g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each $i,\ j$. (The converse is always true.) Some properties of Armendariz rings were given in [1, 2, 5, 6, 11]. Throughout this paper R denotes an associative ring with identity. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. The name *Armendariz ring* was chosen because Armendariz [2] (Lemma 1) had noted that a *reduced* ring (i.e., $a^2 = 0$ implies $a = 0$) satisfies this condition. Zhongkui Liu and Renyu Zhao [9] studied a generalization of Armendariz ring, which is called weak Armendariz ring. A ring R is called *weak Armendariz* if whenever $f(x) = a_0 + a_1x + \ldots + a_mx^m,\ g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x]$, with $a_i, b_j \in R$ satisfy $f(x)g(x) = 0$, then a_ib_j is a nilpotent element of R for each i, j. They have shown that, if R is a semicommutative ring, then the ring $R[x]$ and the ring $\dfrac{R[x]}{(x^n)}$, are weak Armendariz. For an endomorphism α of a ring R, Hong, Kim, and Kwak [3] called R an α-skew Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \ldots + a_mx^m,\ g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$, then $a_i\alpha^i(b_j) = 0$ for each i and j.

Recall from [10] that a one-sided ideal I of a ring R has the *insertion of factors property* (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a,\ b \in R$. Observe that every *completely semiprime* ideal (i.e., $a^2 \in I$ implies $a \in I$) of R has the IFP (or R is semicommutative).

For any positive integer n, we study in this paper the relationship between ideals of R which are weak α-skew Armendariz and some ideals of the ring

$$R_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \ldots & a_{1n} \\ 0 & a & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix} \middle|\ a, a_{ij} \in R, \quad \text{for all} \quad i, j \right\},$$

the n-by-n upper triangular matrix ring over R and the ring $\dfrac{R[x]}{(x^n)}$, where (x^n) is the ideal generated by x^n. Also we show that, if I an ideal of R, then I is a weak α-skew Armendariz if and only if $I[x]$ is a weak α-skew Armendariz ideal.

2. On weak α-skew Armendariz ideals. For an ideal I of R put

$$\sqrt{I} = \{a \in R \mid a^n \in I \ \text{ for some non-negative integer } n\}.$$

Definition 2.1. *Let α be an endomorphism of a ring R, an ideal I of R is said to be weak α-skew Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \ldots + a_mx^m$, $g(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x]$ satisfy $f(x)g(x) \in I[x]$ then $a_i\alpha^i(b_j) \in \sqrt{I}$ for all i, j.*

Clearly, if $I = 0$ is a weak α-skew Armendariz ideal, then R is a weak α-skew Armendariz ring.

It is well-known that for a ring R and any positive integer $n \geq 2$,

$$\frac{R[x]}{(x^n)} \cong \left\{ \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ 0 & a_0 & \ldots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_0 \end{pmatrix} \middle| \, a_i \in R, \quad i = 0, 1, \ldots, n-1 \right\},$$

where (x^n) is the ideal of $R[x]$ generated by x^n.

We introduced a weak α-skew Armendariz ideal in the following example.

***Example* 2.1.** Let R be a α-skew Armendariz ring and consider

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a,\ b \in R \right\}.$$

It is clear that $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| \, b \in R \right\}$ is the ideal of S. Let $f(x) = A_0 + A_1x + \ldots + A_nx^n$, $g(x) = B_0 + B_1x + \ldots + B_mx^m \in S[x]$, where $A_i = \begin{pmatrix} a_{0i} & a_{1i} \\ 0 & a_{0i} \end{pmatrix}$, $B_j = \begin{pmatrix} b_{0j} & b_{1j} \\ 0 & b_{0j} \end{pmatrix}$ for $i = 0, \ldots, n,\ j = 0, \ldots, m$ such that $f(x)g(x) \in I[x]$. Let

$$f(x) = \begin{pmatrix} \alpha_0(x) & \alpha_1(x) \\ 0 & \alpha_0(x) \end{pmatrix}, \qquad g(x) = \begin{pmatrix} \beta_0(x) & \beta_1(x) \\ 0 & \beta_0(x) \end{pmatrix},$$

$$\alpha_0(x) = a_{00} + a_{01}x + \ldots + a_{0n}x^n, \qquad \beta_0(x) = b_{00} + b_{01}x + \ldots + b_{0m}x^m.$$

Since $f(x)g(x) \in I[x]$ thus $\alpha_0(x)\beta_0(x) = 0$, also R is an α-skew Armendariz ring and hence $a_{0i}\alpha^i(b_{0j}) = 0$ for all $i = 0, \ldots, n,\ j = 0, \ldots, m$. Thus $A_i\alpha^i(B_j) \in I$ for all $i = 0, \ldots, n$, $j = 0, \ldots, m$. Therefore I is a weak α-skew Armendariz ideal.

Lemma 2.1. *Let R be a ring and $n \geq 2$ a positive integer. Let $I_0, I_1, \ldots, I_{n-1}$ are ideals of R, such that $I_i \subseteq I_{i+1}$, $i = 0, 1, \ldots, n-2$. Then*

$$J = \left\{ \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ 0 & a_0 & \ldots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_0 \end{pmatrix} \middle| \, a_i \in I_i, \quad i = 0, 1, \ldots, n-1 \right\}$$

is an ideal of $\dfrac{R[x]}{(x^n)}$.

Proof. It is straightforward.

We note that, in Proposition 2.1 and Theorem 2.1, I_0 and J are ideals that mentioned in Lemma 2.1.

Proposition 2.1. *Let*

$$A_i = \begin{pmatrix} a_0^i & a_1^i & \ldots & a_{n-1}^i \\ 0 & a_0^i & \ldots & a_{n-2}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_0^i \end{pmatrix}, \qquad B_j = \begin{pmatrix} b_0^j & b_1^j & \ldots & b_{n-1}^j \\ 0 & b_0^j & \ldots & b_{n-2}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_0^j \end{pmatrix} \in \frac{R[x]}{(x^n)}$$

such that $(a_0^i \alpha^i(b_0^j))^k \in I_0$ *for any* $i,\ j$ *and some integer* k. *Then* $(A_i \alpha^i(B_j))^{nk} \in J$.

Proof. We proceed by induction on n. Let $n = 2$. For a positive integer k, $(A_i \alpha^i(B_j))^k = \begin{pmatrix} (a_0^i \alpha^i(b_0^j))^k & c \\ 0 & (a_0^i \alpha^i(b_0^j))^k \end{pmatrix}$ and that

$$(A_i \alpha^i(B_j))^{2k} = \begin{pmatrix} (a_0^i \alpha^i(b_0^j))^{2k} & (a_0^i \alpha^i(b_0^j))^k c + c(a_0^i \alpha^i(b_0^j))^k \\ 0 & (a_0^i \alpha^i(b_0^j))^{2k} \end{pmatrix}.$$

Hence $(A_i \alpha^i(B_j))^{2k} \in J$, since $\left(a_0^i \alpha^i(b_0^j)\right)^{2k},\ (a_0^i \alpha^i(b_0^j))^k c + c(a_0^i \alpha^i(b_0^j))^k \in I_0$. Now, we have

$$A_i = \begin{pmatrix} a_0^i & a_1^i & \ldots & a_{n-1}^i \\ 0 & a_0^i & \ldots & a_{n-2}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_0^i \end{pmatrix} \in \frac{R[x]}{(x^n)}$$

and

$$B_j = \begin{pmatrix} b_0^j & b_1^j & \ldots & b_{n-1}^j \\ 0 & b_0^j & \ldots & b_{n-2}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_0^j \end{pmatrix} \in \frac{R[x]}{(x^n)},$$

such that $(a_0^i \alpha^i(b_0^j))^k \in I_0$ for some integer k. Consider

$$(A_i\alpha^i(B_j))^k = \begin{pmatrix} (a_0^i\alpha^i(b_0^j))^k & c_1 & \ldots & c_{n-1} \\ 0 & (a_0^i\alpha^i(b_0^j))^k & \ldots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (a_0^i\alpha^i(b_0^j))^k \end{pmatrix} \in J$$

and

$$(A_i\alpha^i(B_j))^{(n-1)k} = \begin{pmatrix} (a_0^i\alpha^i(b_0^j))^{(n-1)k} & d_1 & \ldots & d_{n-1} \\ 0 & (a_0^i\alpha^i(b_0^j))^{(n-1)k} & \ldots & d_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (a_0^i\alpha^i(b_0^j))^{(n-1)k} \end{pmatrix} \in J.$$

By the induction hypothesis all d_i's, except d_{n-1}, are in I_0. Let $x = (a_0^i\alpha^i(b_0^j))^k d_{n-1} + c_1 d_{n-2} + \ldots$ $\ldots + c_{n-1}(a_0^i\alpha^i(b_0^j))^{(n-1)k}$. Hence

$$(A_i\alpha^i(B_j))^{nk} = \begin{pmatrix} (a_0^i\alpha^i(b_0^j))^{nk} & y_1 & \ldots & x \\ 0 & (a_0^i\alpha^i(b_0^j))^{nk} & \ldots & y_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (a_0^i\alpha^i(b_0^j))^{nk} \end{pmatrix} \in J,$$

since $(a_0^i\alpha^i(b_0^j))^{nk},\ x$ all y_i's are in I_0.

Proposition 2.1 is proved.

Theorem 2.1. *I_0 is a weak α-skew Armendariz ideal if and only if J is a weak α-skew Armendariz ideal.*

Proof. $(\Rightarrow)$ Let $f(y) = A_0 + A_1 y + \ldots + A_m y^m$, $g(y) = B_0 + B_1 y + \ldots + B_t y^t \in \dfrac{R[x]}{(x^n)}[y]$, such that $f(y)g(y) \in J[y]$. Let

$$A_i = \begin{pmatrix} a_0^i & a_1^i & \ldots & a_{n-1}^i \\ 0 & a_0^i & \ldots & a_{n-2}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_0^i \end{pmatrix}, \qquad B_j = \begin{pmatrix} b_0^j & b_1^j & \ldots & b_{n-1}^j \\ 0 & b_0^j & \ldots & b_{n-2}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_0^j \end{pmatrix}$$

for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, t$. Let $f_0 = a_0^0 + a_0^1 y + \ldots + a_0^m y^m$ and $g_0 = b_0^0 + b_0^1 y + \ldots + b_0^t y^t$. Then $f_0 g_0 \in I_0[y]$. Since I_0 is weak α-skew Armendariz, there exists $k > 0$, such that $(a_0^i \alpha^i b_0^j)^k \in I_0$ for each i, j. Then $(A_i\alpha^i(B_j))^{nk} \in J$ for all i, j, by Proposition 2.1. Therefore J is weak α-skew Armendariz.

$(\Leftarrow)$ Clear.

Theorem 2.1 is proved.

It can be simply proved if I be an ideal of ring R, then $T_n(I)$ will also be an ideal of ring $T_n(R)$, where $T_n(I)$ is an upper triangle matrix. By the following example we show that $T_2(p\mathbb{Z})$ is a weak α-skew Armendariz ideal.

***Example* 2.2.** Let $p\mathbb{Z}$ be a prime ideal of $\mathbb{Z}$ and $\alpha\colon p\mathbb{Z} \to p\mathbb{Z}$ be an endomorphism. Then $T_2(p\mathbb{Z})$ is a weak α-skew Armendariz ideal.

Let

$$\gamma(x) = \sum_{i=0}^{n} \begin{pmatrix} \gamma_0^i & \gamma_1^i \\ 0 & \gamma_2^i \end{pmatrix} x^i, \qquad \beta(x) = \sum_{j=0}^{n} \begin{pmatrix} \beta_0^j & \beta_1^j \\ 0 & \beta_2^j \end{pmatrix} x^j \in T_2(\mathbb{Z})[x],$$

such that $\gamma(x)\beta(x) \in T_2(p\mathbb{Z})[x]$. Let

$$\gamma(x) = \begin{pmatrix} \gamma_0(x) & \gamma_1(x) \\ 0 & \gamma_2(x) \end{pmatrix}, \qquad \beta(x) = \begin{pmatrix} \beta_0(x) & \beta_1(x) \\ 0 & \beta_2(x) \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \gamma_0(x) & \gamma_1(x) \\ 0 & \gamma_2(x) \end{pmatrix} \begin{pmatrix} \beta_0(x) & \beta_1(x) \\ 0 & \beta_2(x) \end{pmatrix} \in T_2(p\mathbb{Z})[x],$$

and hence we have

$$\gamma_0(x)\beta_0(x) \in p\mathbb{Z}[x],$$

$$\gamma_0(x)\beta_1(x) + \gamma_1(x)\beta_2(x) \in p\mathbb{Z}[x],$$

$$\gamma_2(x)\beta_2(x) \in p\mathbb{Z}[x].$$

Since $p\mathbb{Z}[x]$ is a prime ideal of $\mathbb{Z}$, two cases happen for polynomials,

Case 1. $\gamma_0(x), \gamma_1(x), \gamma_2(x) \in p\mathbb{Z}[x]$, therefore

$$\begin{pmatrix} \gamma_0^i & \gamma_1^i \\ 0 & \gamma_2^i \end{pmatrix} \alpha^i \begin{pmatrix} \beta_0^j & \beta_1^j \\ 0 & \beta_2^j \end{pmatrix} \in T_2(p\mathbb{Z}).$$

Case 2. $\gamma_0(x), \beta_2(x) \in p\mathbb{Z}[x]$, therefore

$$\begin{pmatrix} \gamma_0^i & \gamma_1^i \\ 0 & \gamma_2^i \end{pmatrix} \alpha^i \begin{pmatrix} \beta_0^j & \beta_1^j \\ 0 & \beta_2^j \end{pmatrix} \in T_2(p\mathbb{Z}).$$

Thus $T_2(p\mathbb{Z})$ is a weak α-skew Armendariz ideal.

Let α be an endomorphism of a ring R, $M_n(R)$ be the $n \times n$ full matrix ring over R and $\overline{\alpha} : M_n(R) \longrightarrow M_n(R)$ defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. Then $\overline{\alpha}$ is an endomorphism of $M_n(R)$.

Theorem 2.2. *I_0 is a weak α-skew Armendariz ideal if and only if J is a weak $\overline{\alpha}$-skew Armendariz ideal.*

Proof. $(\Rightarrow)$ Let $f(y) = A_0 + A_1 y + \ldots + A_p y^p$, $g(y) = B_0 + B_1 y + \ldots + B_q y^q \in \dfrac{R[x]}{(x^n)}[y;\overline{\alpha}]$ satisfying $f(y)g(y) \in J[y]$, where

$$A_i = \begin{pmatrix} a^i & a^i_{12} & a^i_{1n} & \ldots & a^i_{1n} \\ 0 & a^i & a^i_{23} & \ldots & a^i_{2n} \\ 0 & 0 & a^i & \ldots & a^i_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a^i \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} b^i & b^i_{12} & b^i_{13} & \ldots & b^i_{1n} \\ 0 & b^i & b^i_{23} & \ldots & b^i_{2n} \\ 0 & 0 & b^i & \ldots & b^i_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & b^i \end{pmatrix}$$

for $i = 0, 1, \ldots, p,\ j = 0, 1, \ldots, q$. Let $f_0 = a^0_0 + a^1_0 y + \ldots + a^p_0 y^p$ and $g_0 = b^0_0 + b^1_0 y + \ldots + b^q_0 y^q$. Then $f_0 g_0 \in I_0[y]$. Since I_0 is weak α-skew Armendariz, there exists $k > 0$, such that $(a^i_0 \alpha^i(b^j_0))^k \in I_0$ for each $i,\ j$. Then $(A_i \overline{\alpha}^i(B_j))^{nk} \in J$ for all $i,\ j$, by Proposition 2.1 and $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. Therefore J is a weak $\overline{\alpha}$-skew Armendariz ideal.

$(\Leftarrow)$ Clear.

Theorem 2.2 is proved.

For the case of weak α-skew Armendariz ideal, we have the following result.

Theorem 2.3. *Let α be an endomorphism of a ring R and $\alpha^t = 1_R$ for some positive integer t. Then I is a weak α-skew Armendariz ideal if and only if $I[x]$ is a weak α-skew Armendariz ideal.*

Proof. $(\Rightarrow)$ Assume that I is a weak α-skew Armendariz ideal. Suppose that $p(y) = f_0(x) + f_1(x)y + \ldots + f_m(x)y^m$ and $q(y) = g_0(x) + g_1(x)y + \ldots + g_n(x)y^n$ are in $R[x][y;\alpha]$ with $p(y)q(y) \in I[x][y;\alpha]$. We also let $f_i(x) = a_{i0} + a_{i1}x + \ldots + a_{i\omega_i}x^{\omega_i}$ and $g_j(x) = b_{j0} + b_{j1}x + \ldots + b_{jv_j}x^{v_j}$ for any $0 \le i \le m$, $0 \le j \le n$, where $a_{i0},\ a_{i1}, \ldots,\ a_{i\omega_i},\ b_{j0},\ b_{j1}, \ldots,\ b_{jv_j} \in R$. We claim that $f_i(x)\alpha^i(g_j(x)) \in \sqrt{I[x]}$ for all $0 \le i \le m$ and $0 \le j \le n$. Take a positive integer k such that $k > \deg(f_0(x)) + \deg(f_1(x)) + \ldots + \deg(f_m(x)) + \deg(g_0(x)) + \deg(g_1(x)) + \ldots + \deg(g_n(x))$, where the degree is as a polynomials in $R[x]$ and the degree of the zero polynomial is taken to be 0. Since $p(y)q(y) \in I[x][y;\alpha]$, we have

$$\begin{gathered} f_0(x)g_0(x) \in I[x], \\ f_0(x)g_1(x) + f_1(x)\alpha(g_0(x)) \in I[x], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_m(x)\alpha^m(g_n(x)) \in I[x]. \end{gathered} \tag{1}$$

Now put

$$\begin{gathered} f(x) = f_0(x^t) + f_1(x^t)x^{tk+1} + f_2(x^t)x^{2tk+2} + \ldots + f_m(x^t)x^{mtk+m}, \\ g(x) = g_0(x^t) + g_1(x^t)x^{tk+1} + g_2(x^t)x^{2tk+2} + \ldots + g_n(x^t)x^{ntk+n}. \end{gathered} \tag{2}$$

Note that $\alpha^t = 1_R$, then $f(x)g(x) = f_0(x^t)g_0(x^t) + (f_0(x^t)g_1(x^t) + f_1(x^t)\alpha(g_0(x^t)))x^{tk+1} + \ldots + f_m(x^t)\alpha^m(g_n(x^t))x^{(m+n)(tk+1)}$. Using (1) and $\alpha^t = 1_R$, we have $f(x)g(x) \in I[x;\alpha]$. On the other hand, from (2) we have

$$f(x)g(x)=\Big(a_{00}+a_{01}x^{t}+\ldots+a_{0\omega_0}x^{\omega_0 t}+a_{10}x^{tk+1}+a_{11}x^{tk+t+1}+\ldots$$

$$\ldots+a_{1\omega_1}x^{tk+\omega_1 t+1}+\ldots+a_{m0}x^{mtk+m}+a_{m1}x^{mtk+t+m}+\ldots+a_{m\omega_m}x^{mtk+\omega_m t+m}\Big)\times$$

$$\times\Big(b_{00}+b_{01}x^{t}+\ldots+b_{0\upsilon_0}x^{\upsilon_0 t}+b_{10}x^{tk+1}+b_{11}x^{tk+t+1}+\ldots+b_{1\upsilon_1}x^{tk+\upsilon_1 t+1}+\ldots$$

$$\ldots+b_{n0}x^{ntk+n}+b_{n1}x^{ntk+t+n}+\ldots+b_{n\upsilon_n}x^{ntk+\upsilon_n t+n}\Big)\in I[x;\alpha].$$

Since I is a weak α-skew Armendariz ideal and $\alpha^t=1_R$, so $a_{iu}\alpha^i(b_{jv})=a_{iu}\alpha^{itk+ut+i}(b_{jv})\in\sqrt{I}$ for all $0\le i\le m$ and $0\le j\le n$, $u\in\{0,1,\ldots,\omega_0,\ldots,\omega_m\}$, $v\in\{0,1,\ldots,\upsilon_0,\ldots,\upsilon_n\}$. So we have $f_i(x^t)\alpha^i(g_j(x^t))\in\sqrt{I[x]}$ for all $0\le i\le m$ and $0\le j\le n$. Now it is easy to see that $f_i(x)\alpha^i(g_j(x))\in\sqrt{I[x]}$ for all $0\le i\le m$ and $0\le j\le n$. Hence $I[x]$ is weak α-skew Armendariz. ($\Leftarrow$) Obviously, if $I[x]$ is weak α-skew Armendariz, then I is weak α-skew Armendariz.

Theorem 2.3 is proved.

Using Theorem 2.3, we have the following result.

***Corollary* 2.1.** *Let R be a ring. Then R is weak α-skew Armendariz if and only if $R[x]$ is weak α-skew Armendariz.*

Before stating Proposition 2.3, we need the following.

Proposition 2.2. *Suppose that there exists a classical right quotient ring Q of a ring R consisting of central elements. If I is IFP, then QI is IFP.*

Proof. Let $\alpha\beta\in QI$ with $\alpha=u^{-1}a$, $\beta=\upsilon^{-1}b$ in Q such that, $u,\ \upsilon\in R$ and $a,b\in R$. Since Q is contained in the center of R, we have $(u\upsilon)^{-1}ab=(u^{-1}\upsilon^{-1})ab=u^{-1}a\upsilon^{-1}b=\alpha\beta\in QI$, so $ab\in I$, and hence $arb\in I$ for all $r\in R$ because I is IFP. Now for $\gamma=\omega^{-1}r$ with $\omega\in R$ and $r\in R$, $\alpha\gamma\beta=(u\omega\upsilon)^{-1}arb\in QI$. Therefore QI is IFP.

Proposition 2.2 is proved.

A ring R is called *right Ore* if given $a,\ b\in R$ with b regular there exist $a_1,\ b_1\in R$ with b_1 regular such that $ab_1=ba_1$. It is a well-known fact that R is a right Ore ring if and only if there exists a classical right quotient ring of R.

Let α be an automorphism of a ring R. Suppose that there exists the classical left quotient Q of R. Then for any $b^{-1}a\in Q$, where $a,\ b\in R$ with b regular the induced map $\overline{\alpha}\colon Q(R)\to Q(R)$ defined by $\overline{\alpha}(b^{-1}a)=(\alpha(b))^{-1}\alpha(a)$ is also an automorphism.

Proposition 2.3. *Suppose that there exists the classical left quotient Q of a ring R. If I is IFP, then I is weak α-skew Armendariz if and only if QI is weak $\overline{\alpha}$-skew Armendariz.*

Proof. Suppose that I is weak α-skew Armendariz. Let $f(x)=s_0^{-1}a_0+s_1^{-1}a_1x+\ldots+s_m^{-1}a_mx^m$ and $g(x)=t_0^{-1}b_0+t_1^{-1}b_1x+\ldots+t_n^{-1}b_nx^n\in QI[x;\overline{\alpha}]$ such that $f(x)g(x)\in QI[x]$. Let C be a left denominator set. There exist $s,t\in C$ and $a'_i,b'_j\in R$ such that $s_i^{-1}a_i=s^{-1}a'_i$ and $t_j^{-1}b_j=t^{-1}b'_j$ for $i=0,1,\ldots,m$ and $j=0,1,\ldots,n$. Then $s^{-1}(a'_0+a'_1x+\ldots+a'_mx^m)t^{-1}(b'_0+$ $+b'_1x+\ldots+b'_nx^n)\in QI[x]$. It follows that $(a'_0+a'_1x+\ldots+a'_mx^m)t^{-1}(b'_0+b'_1x+\ldots+b'_nx^n)\in$ $\in QI[x]$. Thus $(a'_0t^{-1}+a'_1(\alpha(t))^{-1}x+\ldots+a'_m(\alpha^m(t))^{-1}x^m)(b'_0+b'_1x+\ldots+b'_nx^n)\in QI[x]$. For $a'_i(\alpha^i(t))^{-1}$, $i=0,1,\ldots,n$, there exist $t'\in C$ and $a''_i\in R$ such that $a'_i(\alpha^i(t))^{-1}=t'^{-1}a''_i$. Hence

$t'^{-1}(a''_0 + a''_1x + \ldots + a''_mx^m)(b'_0 + b'_1x + \ldots + b'_nx^n) \in QI[x]$. We have that $(a''_0 + a''_1x + \ldots \ldots + a''_mx^m)(b'_0 + b'_1x + \ldots + b'_nx^n) \in I[x]$. Since I is weak α-skew Armendariz, so $a''_i\alpha^i(b'_j) \in \sqrt{I}$ for all i and j. Suppose that $(a''_i\alpha^i(b'_j))^{n_{ij}} \in I$. Since I is IFP, QI is IFP. Then $(t'^{-1}(a''_i\alpha^i(b'_j)))^{n_{ij}} \in \in QI$. So $(a'_i\overline{\alpha}^i(t^{-1}b'_j))^{n_{ij}} = (a'_i(\alpha^i(t))^{-1}\alpha^i(b'_j))^{n_{ij}} = ((t'^{-1}a''_i)\alpha^i(b'_j))^{n_{ij}} \in QI$. Similarly we have $(s_i^{-1}a'_i)(\overline{\alpha}^i(t_j^{-1}b'_j))^{n_{ij}} = (s^{-1}a'_i)(\overline{\alpha}^i(t^{-1}b'_j))^{n_{ij}} \in QI$. Therefore QI is weak $\overline{\alpha}$-skew Armendariz. The converse is clear.

Proposition 2.3 is proved.

We study the relationship between ideals of R which are weak α-skew Armendariz with some ideals of the ring $R_n(R)$.

Lemma 2.2. *Let I, I_{ij} be ideals of R such that $I \subseteq I_{ij} \subseteq I_{is}$ for $1 \le i < j \le s \le n$, and $I_{pq} \subseteq I_{lq}$ for $q = 3, \ldots, n$, $2 \le l \le p \le n$. Then*

$$J = \left\{ \begin{pmatrix} a & a_{12} & \ldots & a_{1n} \\ 0 & a & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix} \middle| \, a \in I, \quad a_{ij} \in I_{ij} \right\}$$

is an ideal of $R_n(R)$.

Proof. It is straightforward.

In Proposition 2.4 and Theorem 2.4, I and J are ideals that mentioned in Lemma 2.2.

Proposition 2.4. *Let*

$$A_i = \begin{pmatrix} a^i & a_1^i & \ldots & a_{n-1}^i \\ 0 & a^i & \ldots & a_{n-2}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a^i \end{pmatrix}, \qquad B_j = \begin{pmatrix} b^j & b_1^j & \ldots & b_{n-1}^j \\ 0 & b^j & \ldots & b_{n-2}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b^j \end{pmatrix} \in R_n(R)$$

such that $(a_0^i\alpha^i(b_0^j))^k \in I$ for any i, j and some integer k. Then $(A_i\alpha^i(B_j))^{nk} \in J$.

Proof. We proceed by induction on n. Let $n = 2$. For a positive integer k,

$$(A_i\alpha^i(B_j))^k = \begin{pmatrix} (a^i\alpha^i(b^j))^k & c \\ 0 & (a^i\alpha^i(b^j))^k \end{pmatrix}$$

and that

$$(A_i\alpha^i(B_j))^{2k} = \begin{pmatrix} (a^i\alpha^i(b^j))^{2k} & (a^i\alpha^i(b^j))^k c + c(a^i\alpha^i(b^j))^k \\ 0 & (a^i\alpha^i(b^j))^{2k} \end{pmatrix}.$$

Hence $(A_i\alpha^i(B_j)) \in J$, since $\left(a^i\alpha^i(b^j)\right)^{2k}$, $(a^i\alpha^i(b^j))^k c + c(a^i\alpha^i(b^j))^k \in I$. Now, let

$$A_i = \begin{pmatrix} a^i & a_1^i & \ldots & a_{n-1}^i \\ 0 & a^i & \ldots & a_{n-2}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a^i \end{pmatrix} \in R_n(R)$$

and

$$B_j = \begin{pmatrix} b^j & b_1^j & \ldots & b_{n-1}^j \\ 0 & b^j & \ldots & b_{n-2}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b^j \end{pmatrix} \in R_n(R)$$

such that $(a^i\alpha^i(b^j))^k \in I$ for some integer k. Consider

$$(A_i\alpha^i(B_j))^k = \begin{pmatrix} (a^i\alpha^i(b^j))^k & c_1 & \ldots & c_{n-1} \\ 0 & (a^i\alpha^i(b^j))^k & \ldots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (a^i\alpha^i(b^j))^k \end{pmatrix} \in J$$

and

$$(A_i\alpha^i(B_j))^{(n-1)k} = \begin{pmatrix} (a^i\alpha^i(b^j))^{(n-1)k} & b_1 & \ldots & d_{n-1} \\ 0 & (a^i\alpha^i(b^j))^{(n-1)k} & \ldots & d_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (a^i\alpha^i(b^j))^{(n-1)k} \end{pmatrix} \in J,$$

by the induction hypothesis all d_i's, except d_{n-1}, are in I. Let $x = (a^i\alpha^i(b^j))^k d_{n-1} + c_1 d_{n-2} + \ldots$ $\ldots + c_{n-1}(a^i\alpha^i(b^j))^{(n-1)k}$. Hence

$$(A_i\alpha^i(B_j))^{nk} = \begin{pmatrix} (a^i\alpha^i(b^j))^{nk} & y_1 & \ldots & x \\ 0 & (a^i\alpha^i(b^j))^{nk} & \ldots & y_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (a^i\alpha^i(b^j))^{nk} \end{pmatrix} \in J,$$

since $(a^i\alpha^i(b^j))^{nk}$, x all y_i's are in I.

Proposition 2.4 is proved.

Theorem 2.4. *I is a weak α-skew Armendariz ideal if and only if J is a weak α-skew Armendariz ideal.*

Proof. ($\Rightarrow$) Let $f(y) = A_0 + A_1 y + \ldots + A_m y^m$, $g(y) = B_0 + B_1 y + \ldots + B_t y^t \in R_n(R)$, such that $f(y)g(y) \in J[y]$. Let

$$A_i = \begin{pmatrix} a_0^i & a_1^i & \ldots & a_{n-1}^i \\ 0 & a_0^i & \ldots & a_{n-2}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_0^i \end{pmatrix}, \qquad B_j = \begin{pmatrix} b_0^j & b_1^j & \ldots & b_{n-1}^j \\ 0 & b_0^j & \ldots & b_{n-2}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_0^j \end{pmatrix}$$

for $i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, t$. Let $f_0 = a_0^0 + a_0^1 y + \ldots + a_0^m y^m$ and $g_0 = b_0^0 + b_0^1 y + \ldots + b_0^t y^t$. Then $f_0 g_0 \in I[y]$. Since I is weak α-skew Armendariz, there exists $k > 0$, such that $(a_0^i \alpha^i b_0^j)^k \in I$ for each $i,\ j$. Then $(A_i \alpha^i(B_j))^{nk} \in J$ for all $i,\ j$, by Proposition 2.4. Therefore J is weak α-skew Armendariz.

($\Leftarrow$) Clear.

Theorem 2.4 is proved.

***Corollary* 2.2.** *A ring R is weak α-skew Armendariz if and only if for any positive integer n, $R_n(R)$ is weak α-skew Armendariz.*

Proof. It follows from Theorem 2.4.

Now, we prove the Theorem 2.4 for $\overline{\alpha}\colon M_n(R) \longrightarrow M_n(R)$.

Theorem 2.5. *I is a weak α-skew Armendariz ideal if and only if J is a weak $\overline{\alpha}$-skew Armendariz ideal.*

Proof. ($\Rightarrow$) Let $f(y) = A_0 + A_1 y + \ldots + A_p y^p,\ g(y) = B_0 + B_1 y + \ldots + B_q y^q \in R_n(R)$ satisfying $f(y)g(y) \in J[y]$, where

$$A_i = \begin{pmatrix} a^i & a_{12}^i & a_{1n}^i & \ldots & a_{1n}^i \\ 0 & a^i & a_{23}^i & \ldots & a_{2n}^i \\ 0 & 0 & a^i & \ldots & a_{3n}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a^i \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} b^i & b_{12}^i & b_{13}^i & \ldots & b_{1n}^i \\ 0 & b^i & b_{23}^i & \ldots & b_{2n}^i \\ 0 & 0 & b^i & \ldots & b_{3n}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & b^i \end{pmatrix}$$

for $i = 0, 1, \ldots, p,\ j = 0, 1, \ldots, q$. Let $f_0 = a_0^0 + a_0^1 y + \ldots + a_0^p y^p$ and $g_0 = b_0^0 + b_0^1 y + \ldots + b_0^q y^q$. Then $f_0 g_0 \in I[y]$. Since I is weak α- skew Armendariz, there exists $k > 0$, such that $(a^i \alpha^i(b^j))^k \in I$ for each $i,\ j$. Then $(A_i \overline{\alpha}^i(B_j))^{nk} \in J$ for all $i,\ j$, by Proposition 2.4, and $\overline{\alpha}(a_{ij}) = (\alpha(a_{ij}))$. Therefore J is a weak $\overline{\alpha}$-skew Armendariz ideal.

($\Leftarrow$) Clear.

Theorem 2.5 is proved.

Theorem 2.6. *Let R be a ring and $I,\ J$ be ideals of R. If $I \subseteq \sqrt{J}$ and $\dfrac{I+J}{I}$ is weak α-skew Armendariz, then J is a weak α-skew Armendariz ideal.*

Proof. Let $f(x) = \Sigma_{i=0}^m a_i x^i,\ g(x) = \Sigma_{j=0}^t b_j x^j \in R[x]$ such that $f(x)g(x) \in J[x]$. Then

$$\left(\Sigma_{i=0}^m \overline{a_i} x^i\right)\left(\Sigma_{j=0}^t \overline{b_j} x^j\right) \in \frac{I+J}{I}[x].$$

Thus $\left(\overline{a_i} \alpha^i \overline{b_j}\right)^{n_{ij}} \in \dfrac{I+J}{I}$ for some positive integer n_{ij}. Hence $(a_i \alpha^i b_j)^{n_{ij}} \in I + J$, and so $\left(a_i \alpha^i b_j\right)^{n_{ij}} \in J$, since $I \subseteq \sqrt{J}$. Therefore J is weak α-skew Armendariz.

Theorem 2.6 is proved.

The following is an immediate corollary of Theorem 2.6.

***Corollary* 2.3.** *Let R be a ring and I an ideal of R such that $\dfrac{R}{I}$ is weak α-skew Armendariz. If $I \subseteq nil(R)$, then R is weak α-skew Armendariz.*

Lemma 2.3. *Let I_{rt} be ideals of R such that $I_{rt} \subseteq I_{rs}$ for $1 \leq r \leq t \leq s \leq n$, and $I_{pq} \subseteq I_{lq}$ for $q = 2, \ldots, n,\ 1 \leq l \leq p \leq n$. Then*

$$J = \left\{ \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{pmatrix} \middle| \ a_{rt} \in I_{rt}, \quad 1 \leq r, \quad t \leq n \right\}$$

is an ideal of $T_n(R)$.

Proof. It is straightforward.

In Corollaries 2.4 and 2.5 I_{rt}'s are ideals that mentioned in Lemma 2.3. By a similar argument as used in the proof of Proposition 2.1 and Theorem 2.1, one can prove Corollaries 2.4 and 2.5.

***Corollary* 2.4.** *Let*

$$A_i = \begin{pmatrix} a^i_{11} & a^i_{12} & \ldots & a^i_{1n} \\ 0 & a^i_{22} & \ldots & a^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a^i_{nn} \end{pmatrix}, \qquad B_j = \begin{pmatrix} b^j_{11} & b^j_{12} & \ldots & b^j_{1n} \\ 0 & b^j_{22} & \ldots & b^j_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b^j_{nn} \end{pmatrix} \in T_n(R)$$

such that $(a^i_{rr}\alpha^i(b^j_{rr}))^k \in I_{rr}$ for some positive integer k and $r = 1, \ldots, n$. Then

$$\left((A_i\alpha^i(B_j))^{2k+1}\right)^{n-1} \in J.$$

***Corollary* 2.5.** *J is a weak α-skew Armendariz ideal if and only if all I_{rr} are weak α-skew Armendariz ideal for $r = 1, \ldots, n$.*

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