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ON INEQUALITIES FOR THE NORMS OF INTERMEDIATE DERIVATIVES OF MULTIPLY MONOTONE FUNCTIONS DEFINED ON A FINITE SEGMENT

ПРО НЕРІВНОСТІ ДЛЯ НОРМ ПРОМІЖНИХ ПОХІДНИХ КРАТНО-МОНОТОННИХ ФУНКЦІЙ, ЩО ЗАДАНІ НА СКІНЧЕННОМУ ВІДРІЗКУ

We study the following modification of the Landau – Kolmogorov problem: Let $k, r \in \mathbb{N}$, $1 \le k \le r-1$, and $p, q, s \in [1,\infty]$. Also let MM^m, $m \in \mathbb{N}$, be the class of nonnegative functions defined on the segment $[0,1]$ whose derivatives of orders $1, 2, \ldots, m$ are nonnegative almost everywhere on $[0,1]$. For every $\delta > 0$, find the exact value of the quantity

$$\omega_{p,q,s}^{k,r}\left(\delta; MM^m\right) := \sup\left\{\left\|x^{(k)}\right\|_q \ : \ x \in MM^m,\ \|x\|_p \le \delta,\ \left\|x^{(r)}\right\|_s \le 1\right\}.$$

We determine the quantity $\omega_{p,q,s}^{k,r}\left(\delta; MM^m\right)$ in the case where $s=\infty$ and $m \in \{r, r-1, r-2\}$. In addition, we consider certain generalizations of the above-stated modification of the Landau – Kolmogorov problem.

Досліджується наступна модифікація задачі Ландау – Колмогорова. Нехай $k, r \in \mathbb{N}$, $1 \le k \le r-1$, $p, q, s \in [1,\infty]$ і MM^m, $m \in \mathbb{N}$, — клас невід'ємних функцій, що задані на відрізку $[0,1]$ та мають майже скрізь на $[0,1]$ невід'ємні похідні порядків $0, 1, \ldots, m$. Для кожного $\delta > 0$ необхідно знайти величину

$$\omega_{p,q,s}^{k,r}\left(\delta; MM^m\right) := \sup\left\{\left\|x^{(k)}\right\|_q \ : \ x \in MM^m,\ \|x\|_p \le \delta,\ \left\|x^{(r)}\right\|_s \le 1\right\}.$$

У даній роботі величину $\omega_{p,q,s}^{k,r}\left(\delta; MM^m\right)$ знайдено у випадку $s=\infty$ та $m \in \{r, r-1, r-2\}$. Також розглянуто деякі узагальнення вказаної модифікації задачі Ландау – Колмогорова.

1. Introduction and statement of the problem. Estimates for the norm of intermediate derivative of function with prescribed bounds on the norm of function itself and the norm of its higher order derivative have various applications in different areas of Mathematics. Sharp estimates of such type are of the most interest. A plenty of remarkable results were obtained in this direction. However, a large number of important questions are still waiting for their solution. For example, sharp estimates for the norm of intermediate derivative of functions given on a finite interval are know only in few exceptional situations. In this paper we find sharp estimates of such type for nonnegative and nondecreasing functions which have several nondecreasing derivatives.

By L_p, $p \in [0,\infty]$, we denote the space of functions $x\colon [0,1] \to \mathbb{R}$ for which the quantity

$$\|x\|_p := \begin{cases} \exp\left(\displaystyle\int_0^1 \ln|x(t)|\, dt\right), & \text{if } \ p = 0, \\ \left(\displaystyle\int_0^1 |x(t)|^p\, dt\right)^{1/p}, & \text{if } \ 0 < p < \infty, \\ \operatorname{ess\,sup}\{|x(t)|\colon t \in [0,1]\}, & \text{if } \ p = \infty, \end{cases}$$

is finite. Obviously, the quantity $\|\cdot\|_p$ is the norm in the space L_p for every $p \in [1,\infty]$. For $r \in \mathbb{N}$, let L_p^r be the space of functions $x\colon [0,1] \to \mathbb{R}$ such that there exists derivative $x^{(r-1)}$ ($x^{(0)} := x$) that is absolutely continuous on $[0,1]$, and $x^{(r)} \in L_p$.

Let numbers $k, r \in \mathbb{N}$, $1 \leq k \leq r-1$, and $p, q, s \in [1, \infty]$ be fixed. *The Landau – Kolmogorov problem* on the interval $[0, 1]$ can be stated as follows.

Problem 1. *For every $\delta > 0$, find*

$$\omega^{k,r}_{p,q,s}(\delta; L^r_s) := \sup \left\{ \left\| x^{(k)} \right\|_q : \ x \in L^r_s, \ \ \|x\|_p \leq \delta, \ \ \|x^{(r)}\|_s \leq 1 \right\}. \qquad (1)$$

Following Steckin [27, 28] we shall call the quantity $\omega^{k,r}_{p,q,s}(\delta; L^r_s)$ *the modulus of continuity* of differential operator of order k on the unit ball $W^r_s := \left\{ x \in L^r_s \colon \|x^{(r)}\|_s \leq 1 \right\}$.

The above stated problem is closely related to the problem of finding sharp additive Kolmogorov type inequalities for derivatives of functions defined on the interval $[0, 1]$. Below we give the rigorous setting of correspondent problem.

Problem 2. *Find the set $\Gamma^{k,r}_{p,q,s}(L^r_s)$ of all pairs (A, B) of positive real numbers which satisfy conditions*:

1) *for every $x \in L^r_s$, there holds inequality*

$$\left\| x^{(k)} \right\|_q \leq A \left\| x \right\|_p + B \|x^{(r)}\|_s; \qquad (2)$$

2) *for every $\varepsilon > 0$, there exists a function $x_\varepsilon \in L^r_s$ such that*

$$\left\| x^{(k)}_\varepsilon \right\|_q > A \left\| x_\varepsilon \right\|_p + (B - \varepsilon) \left\| x^{(r)}_\varepsilon \right\|_s.$$

Remark that the set $\Gamma^{k,r}_{p,q,s}(L^r_s)$ is nonempty for all admissible values of parameters $k, r \in \mathbb{N}$, $1 \leq k \leq r-1$, and $p, q, s \in [1, \infty]$ (see, for instance, [3], Theorem 4.6.2, or [2]).

Let us discuss the connection between Problems 1 and 2. Thus, assume that we were able to evaluate the quantity $\omega^{k,r}_{p,q,s}(\delta; L^r_s)$. Then the set $\Gamma^{k,r}_{p,q,s}(L^r_s)$ can be represented as the union of all pairs (A, B) where $z = A\delta + B$ is the line of support to the graph of function $z = \omega^{k,r}_{p,q,s}(\delta; L^r_s)$. On the other hand, if we know the set $\Gamma^{k,r}_{p,q,s}(L^r_s)$ then we can provide the following upper estimate:

$$\omega^{k,r}_{p,q,s}(\delta; L^r_s) \leq \inf_{(A,B) \in \Gamma^{k,r}_{p,q,s}(L^r_s)} (A\delta + B).$$

Up to nowardays there was not given any complete solution (in the sense of all possible orders k, r of intermediate and upper derivatives) to Problems 1 and 2, even in the case $p = q = s = \infty$. To the best of our knowledge, partial solutions are known only in the following four situations:

1) $p = q = s = \infty$, $r = 2$ – E. Landau [19] (Problem 2) and C. K. Chui, P. W. Smith [12] (Problem 1);

2) $p = q = s = \infty$, $r = 3$ – A. I. Zviagintsev and A. J. Lepin [31], and M. Sato [24] (Problem 1);

3) $p = q = \infty$, $s \in [1, \infty)$, $r = 2$ – Yu. V. Babenko [5] (Problem 2), and V. I. Burenkov and V. A. Gusakov [11] (Problem 1);

4) $p = s = \infty$, $q \in [1, \infty)$, $r = 2$ – B. Bojanov and N. Naidenov [7], and N. Naidenov [21] (Problem 1).

Other results in this direction can be found in books [23, 3] and papers [22, 15, 9, 10, 25, 2, 13, 4, 29].

Remark that in papers [22, 14, 6, 30, 29, 26] it was shown that Problems 1 and 2 could be solved completely if they are considered not on the whole space L_s^r but on some its subset X. In this paper we concern with the study of Problems 1 and 2 as well as their generalizations on the classes of functions that are multiply monotone on the interval $[0,1]$.

In what follows we shall use notation $\mathbb{Z}_+$ for nonnegative integers.

Definition 1. *Let $m \in \mathbb{Z}_+$. A nonnegative and nondecreasing function $x\colon [0,1] \to \mathbb{R}$ is called m-multiply monotone on $[0,1]$ and is written $x \in MM^m$, if its derivatives $x^{(1)}, \ldots, x^{(m-1)}$ are nondecreasing on $[0,1]$.*

For $r, m \in \mathbb{N}$ and $s \in [1,\infty]$, by $L_s^{r,m}$ we denote the subspace of L_s^r consisting of m-multiply monotone functions. Before we state generalizations of Problems 1 and 2 let us introduce some auxiliary definitions.

Definition 2 [16, p. 25]**.** *A function $\Phi : [0,+\infty) \to \mathbb{R}$ is called N-function, if it is continuous, convex and nonnegative on $[0,+\infty)$, and $\Phi(0) = 0$.*

Definition 3 [17, p. 95]**.** *Let Φ be an arbitrary N-function. The Luxembourg norm in the space of continuous functions $x\colon [0,1] \to \mathbb{R}$ is inroduced as follows:*

$$\|x\|_{(\Phi)} := \inf\left\{ \mu > 0\colon \int_0^1 \Phi\left(\frac{|x(t)|}{\mu}\right)\, dt \le 1 \right\}.$$

From the definition it follows that the Luxembourg norm generalizes and in the case $\Phi(t) = t^q$, $q \in [1,\infty)$, coincides with the usual L_q-norm.

For an arbitrary continuous function $x\colon [0,1] \to \mathbb{R}$, we denote by $P(x;\cdot)$ its nonincreasing rearrangement on the interval $[0,1]$ (see [16, p. 17, 18]). The next proposition is the well-known criterion for N-functions (see, for instance, [16], Theorem 3.1.11).

Theorem A. *Let x and y be continuous on $[0,1]$ functions such that for every $t \in [0,1]$,*

$$\int_0^t P(|x|;u)\, du \le \int_0^t P(|y|;u)\, du. \tag{3}$$

Then for an arbitrary N-function Φ,

$$\|x\|_{(\Phi)} \le \|y\|_{(\Phi)}. \tag{4}$$

Conversely, if inequality (4) *holds true for every N-function Φ, then inequality* (3) *holds true as well.*

Now let us state the generalizations of Problems 1 and 2. Fix numbers $k, r \in \mathbb{N}$, $1 \le k \le r-1$, $s \in [1,\infty]$, $p \in [0,\infty]$ and N-function Φ. Let also $X \subset L_s^r$ be a given class of functions.

Problem 3. *For every $\delta > 0$, find*

$$\omega_{p,\Phi,s}^{k,r}(\delta; X) := \sup\left\{ \left\|x^{(k)}\right\|_{(\Phi)} :\ x \in X,\ \ \|x\|_p \le \delta,\ \ \|x^{(r)}\|_s \le 1 \right\}.$$

Problem 4. *Find the set $\Gamma_{p,\Phi,s}^{k,r}(X)$ of all pairs (A, B) of positive real numbers which satisfy conditions*:

1) *for every $x \in X$, there holds inequality*

$$\left\|x^{(k)}\right\|_{(\Phi)} \le A\left\|x\right\|_p + B\|x^{(r)}\|_s; \tag{5}$$

2) *for every* $\varepsilon > 0$, *there exists a function* $x_\varepsilon \in X$ *such that*

$$\left\|x_\varepsilon^{(k)}\right\|_{(\Phi)} > A\left\|x_\varepsilon\right\|_p + (B-\varepsilon)\left\|x_\varepsilon^{(r)}\right\|_s.$$

Here we also study one more problem which is connected with Problems 1 – 4. Let $n \in \mathbb{N}$ and $\mathcal{P}^n$ be the set of all algebraic polynomials of degree at most n. In addition, let X be an arbitrary subset of L_s^r such that $\mathcal{P}^n \cap X \neq \varnothing$.

Problem 5. *For* $k \in \mathbb{N}$, $1 \le k \le n$, $p \in [0,\infty]$ *and* N*-function* Φ *find the lowest possible constant* $M_{p,\Phi}^{k,n}(X)$ *in inequality*

$$\left\|Q^{(k)}\right\|_{(\Phi)} \le M_{p,\Phi}^{k,n}(X)\|Q\|_p, \quad Q \in \mathcal{P}^n \cap X. \tag{6}$$

Inequality (6) is usually called the Markov – Nikolskii type inequality. For a plenty of interesting and important results concerning the solution of Problem 5 we refer reader to books [16, 20, 8].

In this paper we solve Problems 3 and 4 in the case $s = \infty$ for classes $X = L_\infty^{r,r}$, $X = L_\infty^{r,r-1}$ and partially for the class $X = L_\infty^{r,r-2}$. In addition, we find the lowest possible constant in the Markov – Nikolskii type inequality for $(n-1)$-multiple monotone algebraic polynomials of degree at most n, $n \in \mathbb{N}$.

The paper is organized as follows. In the next section we state the main results of this paper. Section 3 is devoted to proofs of several auxilliary statements. In Section 4 we prove main results of this paper.

2. Main results. For given numbers $n \in \mathbb{N}$ and $c \in (0,1]$, we set

$$e_n(t) := \frac{t^n}{n!} \qquad \text{and} \qquad \varphi_{n;c}(t) := \frac{(t-1+c)_+^n}{n!}, \quad t \in [0,1].$$

According to given definition functions e_n and $\varphi_{n;1}$ are coincide. Let also $e_0 \equiv 1$.

Now we define the following set of indices:

$$I = \big\{(\lambda, c) \in \mathbb{R}_+ \times (0,1] \colon \ \lambda = 0 \ \text{ for every } \ c < 1\big\}, \tag{7}$$

and by Θ_n denote the set of functions $\psi\colon [0,1] \to \mathbb{R}$ represented in the form

$$\psi = \lambda e_{n-1} + \varphi_{n;c}, \quad (\lambda, c) \in I.$$

Evidently, for every $\delta > 0$ and $p \in [0,\infty]$, there exists unique function $\psi = \psi_{n,\delta;p} \in \Theta_n$ such that

$$\|\psi_{n,\delta;p}\|_p = \delta. \tag{8}$$

In some cases the function $\psi_{n,\delta;p}$ can be found explicitly. For instance, if we take $p \in (0,\infty]$ and $\delta \le \dfrac{1}{n!(np+1)^{1/p}}$ then

$$\psi_{n,\delta;p} = \phi_{n;c}, \qquad \text{where} \quad c = \Big(\delta n!(np+1)^{1/p}\Big)^{1/(n+1/p)}.$$

In addition, if $p \in \{1,\infty\}$ then for every $\delta > \dfrac{1}{n!(np+1)^{1/p}}$, we have

$$\psi_{n,\delta;p} = e_n + \frac{n!\,(n+1/p)^{1/p}\,\delta - 1}{n+1/p}\, e_{n-1}.$$

The main results of this paper are given by the the following statements.

Theorem 1. *Let numbers $r \in \mathbb{N}$, $r \geq 2$, $m \in \{r-2, r-1, r\}$, $p \in [0,\infty]$ and N-function Φ be given. Then for every $k \in \mathbb{N}$, $1 \leq k \leq m-1$, and $\delta > 0$,*

$$\omega^{k,r}_{p,\Phi,\infty}\left(\delta; L^{r,m}_\infty\right) = \left\|\psi^{(k)}_{r,\delta;p}\right\|_{(\Phi)}, \tag{9}$$

where the function $\psi_{r,\delta;p}$ is determined by (8). *Moreover,*

$$\omega^{r-1,r}_{p,\Phi,\infty}\left(\delta; L^{r,r-1}_\infty\right) = \omega^{r-1,r}_{p,\Phi,\infty}\left(\delta; L^{r,r}_\infty\right). \tag{10}$$

Theorem 2. *Let $r \in \mathbb{N}$, $r \geq 2$, and $p \in [0,\infty]$. Then for every $\delta > 0$,*

$$\begin{aligned}\omega^{r-2,r}_{p,\infty,\infty}\left(\delta; L^{r,r-2}_\infty\right) &= \omega^{r-2,r}_{p,\infty,\infty}\left(\delta; L^{r,r-1}_\infty\right),\\ \omega^{r-1,r}_{p,\infty,\infty}\left(\delta; L^{r,r-2}_\infty\right) &= \omega^{r-1,r}_{p,\infty,\infty}\left(\delta; L^{r,r-1}_\infty\right),\end{aligned} \tag{11}$$

and in the case $r \geq 3$,

$$\omega^{r-2,r}_{p,1,\infty}\left(\delta; L^{r,r-2}_\infty\right) = \omega^{r-2,r}_{p,1,\infty}\left(\delta; L^{r,r-1}_\infty\right). \tag{12}$$

Theorems 1 and 2 allow us to solve Problem 4 for classes $L^{r,r}_\infty$, $L^{r,r-1}_\infty$ and partially for the class $L^{r,r-2}_\infty$. Before we formulate this solution we firstly solve Problem 5 for multiply monotone algebraic polynomials. For $n, m \in \mathbb{N}$, by $\mathcal{P}^{n,m}$ we denote the set of algebraic polynomials Q of degree at most n which are nonnegative on the interval $[0,1]$ along with their derivatives of all orders up to and including m.

Theorem 3. *Let $n \in \mathbb{N}$, $p \in [0,\infty]$ and N-function Φ be given. Then for every $k \in \mathbb{N}$, $1 \leq k \leq n$, and every algebraic polynomial $Q \in \mathcal{P}^{n,n-1}$ there holds exact inequality*

$$\left\|Q^{(k)}\right\|_{(\Phi)} \leq \left(\left\|e^{(k)}_n\right\|_{(\Phi)} \|e_n\|_p^{-1}\right) \|Q\|_p. \tag{13}$$

Remark that the case when $p = \infty$ and L_∞-norm is taken instead of the Luxembourg norm inequality (13) for polynomials $Q \in \mathcal{P}^{n,n}$ was earlier independently established in papers [18] and [26]. In addition, in paper [26] the cases of when $p \in \{1,\infty\}$ and L_q-norms, $q \in \{1,\infty\}$, are taken instead of the Luxembourg norms were considered.

Now we introduce an auxiliary function. For $k, r \in \mathbb{N}$, $1 \leq k \leq r-1$, $p \in [0,\infty]$, $A \geq \left\|e^{(k)}_{r-1}\right\|_{(\Phi)} \|e_{r-1}\|_p^{-1}$ and N-function Φ, define

$$B^{k,r}_{p,\Phi}(A) := \sup_{\psi \in \Theta_r} \left(\left\|\psi^{(k)}\right\|_{(\Phi)} - A\|\psi\|_p\right). \tag{14}$$

An important property of above-introduced function is given by the following proposition.

Proposition 1. *Let numbers $k, r \in \mathbb{N}$, $1 \le k \le r-1$, $p \in [0,\infty]$ and N-function Φ be given. Then for every $A \ge \left\| e_{r-1}^{(k)} \right\|_{(\Phi)} \|e_{r-1}\|_p^{-1}$, the function $B_{p,\Phi}^{k,r}(A)$ is finite and nonnegative.*

In some certain situations we can provide an explicit formula for the function $B_{p,\Phi}^{k,r}(A)$. For instance, the following proposition holds true.

Proposition 2. *Let $k, r \in \mathbb{N}$, $1 \le k \le r-1$, $p \in [0,1] \cup \{\infty\}$ and $q \in [1,\infty]$. Then for every $A \ge \left\| e_{r-1}^{(k)} \right\|_q \|e_{r-1}\|_p^{-1}$,*

$$B_{p,(\cdot)^q}^{k,r}(A) = \lambda(1-\lambda)^{1/\lambda - 1} \left\| e_r^{(k)} \right\|_q^{1/\lambda} \|e_r\|_p^{1-1/\lambda} A^{1-1/\lambda}, \qquad \lambda = \frac{k - 1/q + 1/p}{r + 1/p}.$$

The solution to Problem 4 is given by the following theorem.

Theorem 4. *Let numbers $r \in \mathbb{N}$, $m \in \{r-2, r-1, r\}$, $p \in [0,\infty]$ and N-function Φ be given. Then for every $k \in \mathbb{N}$, $1 \le k \le r-1$,*

$$\Gamma_{p,\Phi,\infty}^{k,r}\left(L_\infty^{r,m}\right) = \left\{ \left(A, B_{p,\Phi}^{k,r}(A)\right) \colon A \ge \left\| e_{r-1}^{(k)} \right\|_{(\Phi)} \|e_{r-1}\|_p^{-1} \right\}.$$

Futhermore,

$$\Gamma_{p,\Phi,\infty}^{r-1,r}\left(L_\infty^{r,r-1}\right) = \Gamma_{p,\Phi,\infty}^{r-1,r}\left(L_\infty^{r,r}\right).$$

Theorem 5. *Let $r \in \mathbb{N}$, $r \ge 2$, and $p \in [0,\infty]$. Then for every $\delta > 0$,*

$$\Gamma_{p,\infty,\infty}^{r-2,r}\left(L_\infty^{r,r-2}\right) = \Gamma_{p,\infty,\infty}^{r-2,r}\left(L_\infty^{r,r-1}\right),$$

$$\Gamma_{p,\infty,\infty}^{r-1,r}\left(L_\infty^{r,r-2}\right) = \Gamma_{p,\infty,\infty}^{r-1,r}\left(L_\infty^{r,r-1}\right),$$

and in the case $r \ge 3$,

$$\Gamma_{p,1,\infty}^{r-2,r}\left(L_\infty^{r,r-2}\right) = \Gamma_{p,1,\infty}^{r-2,r}\left(L_\infty^{r,r-1}\right).$$

3. Auxilliary results. This section is devoted to several auxiliary statements which will be used to prove main results of this paper. For $r, m \in \mathbb{N}$, we set

$$W_\infty^{r,m} := \left\{ x \in L_\infty^{r,m} \colon \ \|x^{(r)}\|_\infty \le 1 \right\}.$$

Lemma 1. *Let numbers $r \in \mathbb{N}$, $r \ge 2$, $m \in \{r-2, r-1, r\}$ and a function $\psi \in \Theta_r$ be given. Then for every $x \in W_\infty^{r,m}$ and $j = 0, 1, \ldots, m$, the difference $x^{(j)} - \psi^{(j)}$ has at most one sign change on $[0,1]$.*

Proof. To prove the assertion of lemma we use ideas from paper [1]. Let a function $\psi \in \Theta_r$ be fixed. According to its definition there exists the pair $(\lambda, c) \in I$ such that

$$\psi = \lambda e_{r-1} + \phi_{r;c}.$$

Hence, the following equalities hold true:

$$\psi^{(j)}(t) = 0, \qquad j = 0, 1, \ldots, r-2 \quad \text{and} \quad t \in [0, 1-c], \tag{15}$$

$$\psi^{(r)}(t) = \left\| \psi^{(r)} \right\|_\infty = 1, \quad t \in [1-c, 1]. \tag{16}$$

Let us show that for every number $j = 0, 1, \ldots, r-2$ and function $x \in W_{\infty}^{r,m}$, the difference $x^{(j)} - \psi^{(j)}$ has at most one sign change on the interval $[0,1]$. To this end we consider the function

$$g(t) := x(t) - \psi(t), \quad t \in [0,1],$$

and assume to the contrary that $g^{(j)}$ has at least two sign changes on $[0,1]$. Since $j \le r-2$ and $m \ge r-2$ we conclude that the function $x^{(j)}$ is nonnegative on the interval $[0,1]$. Thus, by property (15) we have

$$g^{(j)}(t) = x^{(j)}(t) \ge 0 \quad \text{for every} \quad t \in [0, 1-c]. \tag{17}$$

Now, in view of our assumption there exist points ξ_j, η_j, $1 - c < \xi_j < \eta_j \le 1$, such that

$$g^{(j)}(\xi_j) < 0 \qquad \text{and} \qquad g^{(j)}(\eta_j) > 0. \tag{18}$$

By the Lagrange theorem we obtain from (17) and (18) that there exist points ξ_{j+1}, η_{j+1}, $1 - c < < \xi_{j+1} < \eta_{j+1} < 1$, for which

$$g^{(j+1)}(\xi_{j+1}) = \frac{g^{(j)}(\xi_j) - g^{(j)}(1-c)}{\xi_j - 1 + c} < 0,$$

and

$$g^{(j+1)}(\eta_{j+1}) = \frac{g^{(j)}(\eta_j) - g^{(j)}(\xi_j)}{\eta_j - \xi_j} > 0.$$

If $j+1 < r-1$ then $g^{(j+1)}(t) = x^{(j+1)}(t) \ge 0$ for every $t \in [0, 1-c]$, and the function $g^{(j+1)}$ has at least two sign changes on the interval $[0,1]$. Therefore, we can apply the above arguments to prove that each of functions $g^{(j+1)},\ g^{(j+2)}, \ldots,\ g^{(r-2)}$ has at least two sign changes on $[0,1]$. Moreover, we obtain that there exist points ξ_{r-1}, η_{r-1}, $1 - c < \xi_{r-1} < \eta_{r-1} < 1$, such that

$$g^{(r-1)}(\xi_{r-1}) < 0 \qquad \text{and} \qquad g^{(r-1)}(\eta_{r-1}) > 0.$$

Since $x^{(r-1)}$ is absolutely continuous on $[0,1]$ we obtain

$$\int\limits_{\xi_{r-1}}^{\eta_{r-1}} g^{(r)}(t)\, dt = g^{(r-1)}(\eta_{r-1}) - g^{(r-1)}(\xi_{r-1}) > 0. \tag{19}$$

On the other hand, by the choice of function x we have

$$\int\limits_{\xi_{r-1}}^{\eta_{r-1}} g^{(r)}(t)\, dt = \int\limits_{\xi_{r-1}}^{\eta_{r-1}} x^{(r)}(t)\, dt - \int\limits_{\xi_{r-1}}^{\eta_{r-1}} \psi^{(r)}(t)\, dt \le \Big(\|x^{(r)}\|_{\infty} - 1\Big)\,(\eta_{r-1} - \xi_{r-1}) \le 0,$$

which contradicts to inequality (19).

The case $r - 1 \le j \le m$ of this lemma is trivial.

Lemma 1 is proved.

Lemma 2. *Let* $k, r \in \mathbb{Z}_+$, $0 \le k \le r-1$, $r \ge 2$, $p \in [0, \infty]$ *and* $\psi \in \Theta_r$. *Then for every function* $x \in W^{r,r-2}_\infty$ *such that* $\|x\|_p \le \|\psi\|_p$, *there holds inequality*

$$\left\|x^{(k)}\right\|_\infty \le \left\|\psi^{(k)}\right\|_\infty. \tag{20}$$

Proof. Firstly, we prove the assertion of lemma for $k \le r-3$. Assume to the contrary that

$$\left\|x^{(k)}\right\|_\infty > \left\|\psi^{(k)}\right\|_\infty. \tag{21}$$

Since both functions x and ψ are $(r-2)$-monotone on $[0,1]$, we can rewrite inequality (21) in the following form:

$$x^{(k)}(1) - \psi^{(k)}(1) > 0. \tag{22}$$

At the same time, for every $t \in [0, 1-c]$, we have $x^{(k)}(t) \ge 0 = \psi^{(k)}(t)$. Now let us show that there exists a point $\xi \in (1-c, 1)$ for which $x^{(k)}(\xi) < \psi^{(k)}(\xi)$. Indeed, let

$$x^{(k)}(t) \ge \psi^{(k)}(t) \quad \text{for every} \quad t \in [0,1].$$

Then by the Taylor formula we obtain that for every $t \in [0,1]$,

$$x(t) = x(0) + \ldots + \frac{x^{(k-1)}(0)t^{k-1}}{(k-1)!} + \int\limits_0^t \frac{(t-u)^{k-1}}{(k-1)!} x^{(k)}(u)\, du \ge$$

$$\ge \int\limits_0^t \frac{(t-u)^{k-1}}{(k-1)!} \psi^{(k)}(u)\, du = \psi(t).$$

Hence, in view of inequality (22) and continuity of functions $x^{(k)}$ and $\psi^{(k)}$, we have $x(1) > \psi(1)$. This yields that $\|x\|_p > \|\psi\|_p$ which contradicts to the choice of function x.

Therefore, we have proved that

$$x^{(k)}(1-c) \ge \psi^{(k)}(1-c), \qquad x^{(k)}(1) > \psi^{(k)}(1)$$

and there exists a point $\xi \in (1-c, 1)$ such that

$$x^{(k)}(\xi) < \psi^{(k)}(\xi).$$

This shows that the difference $x^{(k)} - \psi^{(k)}$ has at least two sign changes on the interval $[0,1]$, which is impossible due to Lemma 1. Therefore, inequality (20) holds true for every $k \le r-3$.

Let us now prove inequality (20) for $k = r-2$. Assume to the contrary that

$$\left\|x^{(r-2)}\right\|_\infty > \left\|\psi^{(r-2)}\right\|_\infty. \tag{23}$$

Let $\xi \in [0,1]$ be the point of global maximum of function $x^{(r-2)}$. Let also $(\lambda, c) \in I$ be the pair for which

$$\psi = \lambda e_{r-1} + \phi_{r;c}.$$

From inequality (23) we conclude that

$$x^{(r-2)}(\xi) > \frac{c^2}{2} + \lambda. \tag{24}$$

Let us consider three cases: 1) $\xi = 0$, 2) $\xi = 1$, 3) $\xi \in (0,1)$.

1) Let $\xi = 0$. In this case for every $t \in [0, c]$,

$$x^{(r-2)}(t) > \tau_1(t) := \frac{1}{2}(c-t)^2 + \lambda(1-t). \tag{25}$$

Indeed, by inequality (24) and nonnegativity of $x^{(r-2)}$ we obtain

$$x^{(r-2)}(0) > \frac{c^2}{2} + \lambda = \tau_1(0) \qquad \text{and} \qquad x^{(r-2)}(c) \geq 0 = \tau_1(c).$$

Assume that there exists a point $\eta \in (0, c)$ such that $x^{(r-2)}(\eta) \leq \tau_1(\eta)$. Then by the Lagrange theorem there exist points ξ_1, η_1, $0 < \xi_1 < \eta < \eta_1 < c$, for which

$$x^{(r-1)}(\xi_1) < \tau_1'(\xi_1) \qquad \text{and} \qquad x^{(r-1)}(\eta_1) \geq \tau_1'(\eta_1).$$

Therefore,

$$\int\limits_{\xi_1}^{\eta_1} x^{(r)}(t)\; dt = x^{(r-1)}(\eta_1) - x^{(r-1)}(\xi_1) > \tau_1'(\eta_1) - \tau_1'(\xi_1) =$$

$$= \eta_1 - \xi_1 = \int\limits_{\xi_1}^{\eta_1} \left\|\psi^{(r)}\right\|_\infty dt \geq \int\limits_{\xi_1}^{\eta_1} x^{(r)}(t)\; dt,$$

which is impossible. Consequently, inequality (25) holds true.

Since the function $x^{(r-2)}$ is nonnegative on $[0,1]$, we obtain

$$\left\|x^{(r-3)}\right\|_\infty \geq \left\|x^{(r-2)}\right\|_1 > \int\limits_0^c \tau_1(u)\; du = \frac{c^3}{6} + \lambda\frac{c^2}{2} = \left\|\psi^{(r-3)}\right\|_\infty.$$

However the latter inequality contradicts to inequality (20) with $k = r-3$, which we have already proved.

The second case when $\xi = 1$ can be done similarly.

3. Let $\xi \in (0,1)$. Since $x^{(r-1)}(\xi) = 0$, for every $t \in [0,1]$, we have

$$x^{(r-2)}(t) \geq x^{(r-2)}(\xi) - \frac{1}{2}(t-\xi)^2 > \frac{c^2}{2} + \lambda - \frac{1}{2}(t-\xi)^2.$$

Hence,

$$\left\|x^{(r-3)}\right\|_\infty \geq \left\|x^{(r-2)}\right\|_1 \geq \inf_{\eta\in[0,1-c]} \int\limits_\eta^{\eta+c} x^{(r-2)}(t)\; dt =$$

$$= \frac{c^3}{2} + \lambda c - \frac{1}{2}\int\limits_0^c t^2\, dt > \frac{c^3}{6} + \lambda c = \left\|\psi^{(r-3)}\right\|_\infty,$$

which is impossible.

To finish the proof of lemma, we need to verify that inequality (20) holds true for $k = r-1$. Assume to the contrary that there exists a point $\xi \in [0,1]$ such that

$$|x^{(r-1)}(\xi)| > \lambda + c = \left\|\psi^{(r-1)}\right\|_\infty,$$

where $(\lambda, c) \in I$ is the pair of numbers for which

$$\psi = \lambda e_{r-1} + \phi_{r;c}.$$

Here we have to consider two cases: 1) $x^{(r-1)}(\xi) > 0$ and 2) $x^{(r-1)}(\xi) < 0$.

1. If $x^{(r-1)}(\xi) > 0$ then for every $t \in [0,1]$, we have

$$x^{(r-1)}(t) \ge x^{(r-1)}(\xi) - |\xi - t|.$$

Note that $[\xi - c, \xi] \cap [0, 1-c] \ne \varnothing$. The latter inequality yields that for every $\alpha \in [\xi - c, \xi] \cap [0, 1-c]$,

$$x^{(r-2)}(\alpha + c) \ge x^{(r-2)}(\alpha + c) - x^{(r-2)}(\alpha) = \int\limits_\alpha^{\alpha+c} x^{(r-1)}(t)\, dt \ge$$

$$\ge x^{(r-1)}(\xi)c - \inf_{\eta\in[0,1-c]}\int\limits_\eta^{\eta+c} |\xi - t|\, dt > \lambda c + \frac{c^2}{2} = \left\|\psi^{(r-2)}\right\|_\infty.$$

Hence, we obtain

$$\left\|x^{(r-2)}\right\|_\infty > \left\|\psi^{(r-2)}\right\|_\infty,$$

which is impossible. The case when $x^{(r-1)}(\xi) < 0$ can be studied similarly.

Lemma 2 is proved.

The following statement is a corollary of Lemma 2.

Lemma 3. *Let numbers $k, r \in \mathbb{N}$, $1 \le k \le r-2$, $p \in [0,\infty]$ and a function $\psi \in \Theta_r$ be given. If a function $x \in W_\infty^{r,r-2}$ is such that $\|x\|_p \le \|\psi\|_p$ then*

$$\left\|x^{(k)}\right\|_1 \le \left\|\psi^{(k)}\right\|_1.$$

Proof. Indeed, since the function $x^{(k)}$ is nonnegative on the interval $[0,1]$, inequality (20) shows us that

$$\left\|x^{(k)}\right\|_1 = \int\limits_0^1 x^{(k)}(t)\, dt = x^{(k-1)}(1) - x^{(k-1)}(0) =$$

$$\le x^{(k-1)}(1) = \|x^{(k-1)}\|_\infty \le \|\psi^{(k-1)}\|_\infty =$$

$$= \psi^{(k-1)}(1) = \psi^{(k-1)}(1) - \psi^{(k-1)}(0) = \int\limits_0^1 \psi^{(k)}(t)\, dt = \left\|\psi^{(k)}\right\|_1 .$$

Lemma 3 is proved.

Lemma 4. *Let numbers $r \in \mathbb{N}$, $r \geq 2$, $p \in [0,\infty]$, N-function Φ and a function $\psi \in \Theta_r$ be given. If a function $x \in W_\infty^{r,r-1}$ is such that $\|x\|_p \leq \|\psi\|_p$ then*

$$\left\|x^{(r-1)}\right\|_{(\Phi)} \leq \left\|\psi^{(r-1)}\right\|_{(\Phi)} . \tag{26}$$

Proof. In Lemma 2 we have established that $\|x^{(k)}\|_\infty \leq \|\psi^{(k)}\|_\infty$ for every $k \in \mathbb{N}$, $0 \leq k \leq$ $\leq r-1$. Consequently,

$$\left\|x^{(r-1)}\right\|_\infty \leq \left\|\psi^{(r-1)}\right\|_\infty ,$$

and, futhermore,

$$\left\|x^{(r-1)}\right\|_1 = x^{(r-2)}(1) - x^{(r-2)}(0) \leq \left\|x^{(r-2)}\right\|_\infty \leq \left\|\psi^{(r-2)}\right\|_\infty = \left\|\psi^{(r-1)}\right\|_1 .$$

Now let us prove that the difference $P\left(x^{(r-1)};\cdot\right) - P\left(\psi^{(r-1)};\cdot\right)$ has at most one sign change on the interval $[0,1]$. Indeed, by the choice of function x we have that $\left|x^{(r)}(t)\right| \leq 1$ for almost all $t \in [0,1]$. Hence, $\left|P'\left(x^{(r-1)};t\right)\right| \leq 1$ for almost all $t \in [0,1]$. On the other hand, $P\left(\psi^{(r-1)};t\right) =$ $= \max\{\lambda + c - t; 0\}$. Therefore, the graphs of functions $P\left(x^{(r-1)};\cdot\right)$ and $P\left(\psi^{(r-1)};\cdot\right)$ intersect at most once. This implies that for every $t \in [0,1]$,

$$\int\limits_0^t P\left(x^{(r-1)};u\right)\, du \leq \int\limits_0^t P\left(\psi^{(r-1)};u\right)\, du.$$

Now, we can apply Theorem A and verify the validity of inequality (26).

Lemma 4 is proved.

4. Proofs of main results. ***Proof of Theorem* 1.** Firstly, we prove equality (9). To this end we choose an arbitrary function $x \in W_\infty^{r,m}$ for which $\|x\|_p \leq \delta$. We need to show that

$$\left\|x^{(k)}\right\|_{(\Phi)} \leq \left\|\psi_{r,\delta;p}^{(k)}\right\|_{(\Phi)} .$$

It is clear that functions x and $\psi = \psi_{r,\delta;p}$ satisfy conditions of Lemmas 1, 2 and 3. Hence,

$$\left\|x^{(k)}\right\|_\infty \leq \left\|\psi_{r,\delta;p}^{(k)}\right\|_\infty , \qquad \left\|x^{(k)}\right\|_1 \leq \left\|\psi_{r,\delta;p}^{(k)}\right\|_1 ,$$

and the difference $x^{(k)} - \psi_{r,\delta;p}^{(k)}$ has at most one sign change on $[0,1]$. Since $k \leq m-1$, the functions $x^{(k)}$ and $\psi^{(k)}$ are nondecreasing on $[0,1]$, and we conclude that for every $t \in [0,1]$,

$$P\left(x^{(k)};t\right) = x^{(k)}(1-t) \qquad \text{and} \qquad P\left(\psi_{r,\delta;p}^{(k)};t\right) = \psi_{r,\delta;p}^{(k)}(1-t).$$

This yields that for every $t \in [0,1]$,

$$\int_0^t P\left(x^{(k)};u\right)\,du \le \int_0^t P\left(\psi^{(k)}_{r,\delta;p};u\right)\,du.$$

From the latter inequality and Theorem A we obtain desired inequality (9).

To finish the proof of theorem we note that the validity of equality (10) immediately follows from Lemma 4.

Theorem 1 is proved.

Remark that Theorem 2 is a corollary of Lemmas 1 and 2.

***Proof of Theorem* 3.** Firstly, we prove that

$$\left\|Q^{(n)}\right\|_\infty \le \frac{\|Q\|_p}{\|e_n\|_p}. \tag{27}$$

Indeed, since Q is the polynomial of degree at most n we should consider two cases: 1) $Q^{(n)}(0) > 0$ and 2) $Q^{(n)}(0) < 0$.

1. Assume that $Q^{(n)}(0) > 0$. Taking into account the fact that each of functions $Q, Q', \ldots, Q^{(n-1)}$ is nonnegative on the interval $[0,1]$ we obtain that

$$Q(t) = \sum_{m=0}^{n-1} Q^{(m)}(0)e_m(t) + Q^{(n)}(0)e_n(t) \ge Q^{(n)}(0)e_n(t) = \left\|Q^{(n)}\right\|_\infty e_n(t)$$

for every $t \in [0,1]$. Hence,

$$\|Q\|_p \ge \left\|Q^{(n)}\right\|_\infty \|e_n\|_p,$$

which is inequality (27).

2. Let now $Q^{(n)}(0) < 0$. Since $Q^{(n-1)}$ is nonnegative on $[0,1]$, we conclude that $Q^{(n-1)}(t) \ge$ $\ge Q^{(n)}(0)(t-1)$ for every $t \in [0,1]$. If $n = 1$ then $\|Q\|_p \ge \left\|Q^{(1)}\right\|_\infty \|e_1\|_p$ which gives desired inequality (27). If $n \ge 2$ then for every $t \in [0,1]$, we obtain

$$Q(t) = \sum_{m=0}^{n-2} Q^{(m)}(0)e_m(t) + \int_0^t Q^{(n-1)}(u)e_{n-2}(t-u)\,du \ge$$

$$\ge \int_0^t Q^{(n-1)}(u)e_{n-2}(t-u)\,du \ge \left\|Q^{(n)}\right\|_\infty \int_0^t (1-u)e_{n-2}(t-u)\,du =$$

$$= \frac{\left\|Q^{(n)}\right\|_\infty}{(n-2)!}\int_0^t (1-u)(t-u)^{n-2}\,du \ge \frac{\left\|Q^{(n)}\right\|_\infty}{(n-2)!}\int_0^t u(t-u)^{n-2}\,du =$$

$$= \left\|Q^{(n)}\right\|_\infty e_n(t).$$

This yields that $\|Q\|_p \ge \left\|Q^{(n)}\right\|_\infty \|e_n\|_p$. Therefore, inequality (27) is proved.

Let us turn to the proof of Theorem 3. Since $\left|Q^{(n)}(t)\right| = \left\|Q^{(n)}\right\|_\infty$ for every $t \in [0,1]$, from inequality (27) we obtain that

$$\left\|Q^{(n)}\right\|_{(\Phi)} \le \frac{\|e_n^{(n)}\|_{(\Phi)}}{\|e_n\|_p}\|Q\|_p.$$

This is desired inequality (13) for $k = n$.

Let us now prove inequality (13) for $1 \le k \le n-1$. Consider the polynomial

$$x(t) := \frac{\|e_n\|_p}{\|Q\|_p}Q(t), \quad t \in [0,1].$$

It is clear that $\|x\|_p = \|e_n\|_p$. Moreover, in view of inequality (27) we conclude that $x \in W_\infty^{n,n-1}$. Since $e_n \in \Theta_n$ we can apply the assertion of Theorem 1. This yields

$$\left\|x^{(k)}\right\|_{(\Phi)} = \frac{\|e_n\|_p}{\|Q\|_p}\left\|Q^{(k)}\right\|_{(\Phi)} \le \left\|e_n^{(k)}\right\|_{(\Phi)}.$$

Theorem 3 is proved.

Proof of Proposition **1.** Note that for every $A \ge \left\|e_{r-1}^{(k)}\right\|_{(\Phi)} \|e_{r-1}\|_p^{-1}$,

$$B_{p,\Phi}^{k,r}(A) = \max\{B_1; B_2\},$$

where

$$B_1 := \sup_{c\in(0,1]}\left(\left\|\phi_{r;c}^{(k)}\right\|_{(\Phi)} - A\|\phi_{r;c}\|_p\right),$$

$$B_2 := \sup_{\lambda\ge 0}\left(\left\|e_r^{(k)} + \lambda e_{r-1}^{(k)}\right\|_{(\Phi)} - A\|e_r + \lambda e_{r-1}\|_p\right).$$

Let us show that both quantities B_1 and B_2 are finite. Firstly, we prove that $B_1 < \infty$. Indeed, for every $t \in [0,1]$ and $c \in (0,1]$, we have

$$\phi_{r;c}(t) \le \phi_{r;1}(t) = e_r(t).$$

Hence,

$$B_1 \le \sup_{c\in(0,1]}\left(\left\|e_r^{(k)}\right\|_{(\Phi)} - A\|\phi_{r;c}\|_p\right) \le \left\|e_r^{(k)}\right\|_{(\Phi)} < \infty.$$

Now let us prove that $B_2 < \infty$. Indeed,

$$B_2 = \sup_{\lambda\ge 0}\left(\left\|e_r^{(k)} + \lambda e_{r-1}^{(k)}\right\|_{(\Phi)} - A\|e_r + \lambda e_{r-1}\|_p\right) \le$$

$$\le \sup_{\lambda\ge 0}\left(\left\|e_r^{(k)}\right\|_{(\Phi)} + \lambda\left\|e_{r-1}^{(k)}\right\|_{(\Phi)} - \frac{\left\|e_{r-1}^{(k)}\right\|_{(\Phi)}}{\|e_{r-1}\|_p}\lambda\|e_{r-1}\|_p\right) = \left\|e_r^{(k)}\right\|_{(\Phi)}.$$

The nonnegativity of $B^{k,r}_{p,\Phi}(A)$ follows from inequalities

$$B^{k,r}_{p,\Phi}(A) \ge B_1 \ge \lim_{c\to 0}\left(\left\|\phi^{(k)}_{r;c}\right\|_{(\Phi)} - A\left\|\phi_{r;c}\right\|_p\right) = 0.$$

Proposition 1 is proved.

***Proof of Proposition* 2.** As in the proof of Proposition 1 we see that for every $A \ge \ge \left\|e^{(k)}_{r-1}\right\|_q \|e_{r-1}\|_p^{-1}$,

$$B^{k,r}_{p,(\cdot)^q}(A) = \max\{B_1; B_2\},$$

where the quantities B_1 and B_2 were defined before. In view of the choice of numbers p and q for every two functions $x, y \in L^{r,r-1}_\infty$, we have

$$\|x+y\|_q \le \|x\|_q + \|y\|_q \qquad \text{and} \qquad \|x+y\|_p \ge \|x\|_p + \|y\|_p.$$

Hence,

$$B_2 = \sup_{\lambda\ge 0}\left(\left\|e^{(k)}_r + \lambda e^{(k)}_{r-1}\right\|_q - A\|e_r + \lambda e_{r-1}\|_p\right) \le$$

$$\le \sup_{\lambda\ge 0}\left(\left\|e^{(k)}_r\right\|_q + \lambda\left\|e^{(k)}_{r-1}\right\|_q - A\|e_r\|_p - A\lambda\|e_{r-1}\|_p\right) =$$

$$= \left\|e^{(k)}_r\right\|_q - A\|e_r\|_p \le \sup_{c\in(0,1]}\left(\left\|\phi^{(k)}_{r;c}\right\|_q - A\|\phi_{r;c}\|_p\right) = B_1.$$

Therefore,

$$B^{k,r}_{p,(\cdot)^q}(A) = \sup_{c\in(0,1]}\left(\left\|\phi^{(k)}_{r;c}\right\|_q - A\|\phi_{r;c}\|_p\right) =$$

$$= \sup_{c\in(0,1]}\left(\left\|e^{(k)}_r\right\|_q c^{r-k+1/q} - A\|e_r\|_p c^{r+1/p}\right).$$

Simple calculations show us that the function

$$g(c) := \left\|e^{(k)}_r\right\|_q c^{r-k+1/q} - A\|e_r\|_p c^{r+1/p}, \quad c > 0,$$

achieves its maximum at the point

$$c_0 = \left((1-\lambda)\left\|e^{(k)}_r\right\|_q A^{-1}\|e_r\|_p^{-1}\right)^{\frac{1}{k-1/q+1/p}}.$$

Let us show that $c_0 \le 1$. Indeed, since $A \ge \left\|e^{(k)}_{r-1}\right\|_q \|e_{r-1}\|_p^{-1}$ we have

$$c_0^{k-1/q+1/p} \le (1-\lambda)\frac{\left\|e^{(k)}_r\right\|_q \|e_{r-1}\|_p}{\left\|e^{(k)}_{r-1}\right\|_q \|e_r\|_p}. \tag{28}$$

The generalized Bernoulli inequality states that if $x > -1$ then inequality $(1+x)^\alpha \geq 1 + \alpha x$ holds true for $\alpha \in (-\infty, 0] \cup [1, +\infty)$, and inequality $(1+x)^\alpha \leq 1 + \alpha x$ holds true for $\alpha \in [0,1]$. Applying both inequalities we for obtain

$$\frac{\|e_r^{(k)}\|_q}{\left\|e_{r-1}^{(k)}\right\|_q} = \left(1 - \frac{1}{r-k+1/q}\right)^{1/q} \leq 1 - \frac{1/q}{r-k+1/q} = \frac{r-k}{r-k+1/q},$$

and

$$\frac{\|e_r\|_p}{\|e_{r-1}\|_p} = \left(1 - \frac{1}{r+1/p}\right)^{1/p} \geq 1 - \frac{1/p}{r+1/p} = \frac{r}{r+1/p}.$$

Hence,

$$c_0^{k-1/q+1/p} \leq (1-\lambda)\frac{(r-k)\,(r+1/p)}{r\,(r-k+1/q)} = \frac{r-k}{r} < 1.$$

Therefore,

$$B^{k,r}_{p,(\cdot)^q}(A) = g(c_0) = \lambda(1-\lambda)^{1/\lambda - 1} \left\|e_r^{(k)}\right\|_q^{1/\lambda} \|e_r\|_p^{1-1/\lambda} A^{1-1/\lambda}.$$

Proposition 2 is proved.

***Proof of Theorem* 4.** Let us choose an arbitrary pair of numbers $(A,B) \in \Gamma^{k,r}_{p,\Phi,\infty}\left(L^{r,m}_\infty\right)$. In view of inequality (13) we obtain

$$A \geq \left\|e_{r-1}^{(k)}\right\|_{(\Phi)} \|e_{r-1}\|_p^{-1}.$$

Let us show that

$$B \leq B^{k,r}_{p,\Phi}(A). \tag{29}$$

To this end for every function $x \in L^{r,m}_\infty \setminus \mathcal{P}^{r-1,m}$ we define

$$y(t) := \frac{x(t)}{\|x^{(r)}\|_\infty}, \quad t \in [0,1].$$

Evidently, $y \in W^{r,m}_\infty$. Let $\psi \in \Theta_r$ be the function such that $\|\psi\|_p = \|y\|_p$. Then by Theorem 1 we have

$$\left\|x^{(k)}\right\|_{(\Phi)} = \|x^{(r)}\|_\infty \|y^{(k)}\|_{(\Phi)} \leq \|x^{(r)}\|_\infty \left\|\psi^{(k)}\right\|_{(\Phi)} \leq$$

$$\leq \|x^{(r)}\|_\infty \left[A\|\psi\|_p + B^{k,r}_{p,\Phi}(A)\right] = A\|x\|_p + B^{k,r}_{p,\Phi}(A)\|x^{(r)}\|_\infty.$$

It remains to consider the case when $x \in \mathcal{P}^{r-1,m}$. Since $\|x^{(r)}\|_\infty = 0$ Lemma 1 shows us that

$$\left\|x^{(k)}\right\|_{(\Phi)} \leq \frac{\left\|e_{r-1}^{(k)}\right\|_{(\Phi)}}{\|e_{r-1}\|_p} \|x\|_p \leq A\|x\|_p + B^{k,r}_{p,\Phi}(A)\|x^{(r)}\|_\infty.$$

Hence, inequality (29) holds true according to definition of the set $\Gamma^{k,r}_{p,\Phi,\infty}\left(L^{r,m}_{\infty}\right)$. Futhermore, we have just established that for every $A \geq \left\|e^{(k)}_{r-1}\right\|_{(\Phi)} \|e_{r-1}\|^{-1}_{p}$ there exists $B \geq 0$ such that $(A, B) \in$ $\in \Gamma^{k,r}_{p,\Phi,\infty}\left(L^{r,m}_{\infty}\right)$.

On the other hand,

$$B \geq \sup_{\psi \in \Theta_r} \frac{\left\|\psi^{(k)}\right\|_{(\Phi)} - A\|\psi\|_p}{\left\|\psi^{(r)}\right\|_{\infty}} := B^{k,r}_{p,\Phi}(A).$$

Hence, $B = B^{k,r}_{p,\Phi}(A)$.

Theorem 4 is proved.

Theorem 5 can be proved similarly to Theorem 4.

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