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ON A p-LAPLACIAN SYSTEM WITH CRITICAL HARDY – SOBOLEV EXPONENTS AND CRITICAL SOBOLEV EXPONENTS

ПРО p-ЛАПЛАСОВУ СИСТЕМУ З КРИТИЧНИМИ ПОКАЗНИКАМИ ХАРДІ – СОБОЛЄВА ТА КРИТИЧНИМИ ПОКАЗНИКАМИ СОБОЛЄВА

We consider a quasilinear elliptic system involving the critical Hardy – Sobolev exponent and Sobolev exponent. Using variational methods and analytic techniques, we establish the existence of positive solutions of the system.

Розглянуто квазілінійну еліптичну систему з критичними показниками Харді – Соболєва та Соболєва. Із застосуванням варіаційних методів та аналітичного підходу встановлено існування додатних розв'язків системи.

1. Introduction. The aim of this paper is to establish the existence of nontrivial nonnegative solution to the semilinear elliptic system

$$-\Delta_p u_1 - \mu \frac{|u_1|^{p-2}u_1}{|x|^p} = \frac{1}{p^*} F_{u_1}(u_1, \ldots, u_k) + \frac{|u_1|^{p^*(t)-2}u_1}{|x|^t} + \lambda \frac{|u_1|^{p-2}u_1}{|x|^s}, \quad x \in \Omega,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

$$-\Delta_p u_k - \mu \frac{|u_k|^{p-2}u_k}{|x|^p} = \frac{1}{p^*} F_{u_k}(u_1, \ldots, u_k) + \frac{|u_k|^{p^*(t)-2}u_k}{|x|^t} + \lambda \frac{|u_k|^{p-2}u_k}{|x|^s}, \quad x \in \Omega,$$

$$u_i = 0, \quad 1 \le i \le k, \quad \text{on} \quad \partial\Omega,$$

where $\Delta_p u_i = \operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i)$, $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$, $N \ge 3$, with smooth boundary $\partial\Omega$, $1 < p < N$, $0 \le \mu < \overline{\mu} \triangleq \left(\frac{N-p}{p}\right)^p$, $\lambda > 0$, $0 \le t < p$, $p^*(t) \triangleq \frac{p(N-t)}{N-p}$ is the Hardy – Sobolev critical exponent, $p^* = p^*(0) = \frac{pN}{N-p}$ is the Sobolev critical exponent and $\nabla F(u_1, \ldots, u_n) = (F_{u_1}(u_1, \ldots, u_k), \ldots, F_{u_k}(u_1, \ldots, u_k))$, where $F\colon (\mathbb{R}^+)^k \to \mathbb{R}^+$ are C^1 function with positively homogeneous of degree p^*.

We denote by $D^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm $\left(\int_\Omega |\nabla \cdot|^p dx\right)^{1/p}$.

Problem () is related to the well known Caffarelli – Kohn – Nirenberg inequality in [5],

$$\left(\int_\Omega \frac{|u|^r}{|x|^t} dx\right)^{p/r} \le C_{r,t,p} \int_\Omega |\nabla u|^p dx \quad \text{for all} \quad u \in D^{1,p}(\Omega), \qquad (2)$$

where $p \le r < p^*(t)$. If $t = r = p$, the above inequality becomes the well known Hardy inequality [5, 10, 13]

$$\int_\Omega \frac{|u|^p}{|x|^p} dx \le \frac{1}{\overline{\mu}} \int_\Omega |\nabla u|^p dx \qquad \text{for all} \qquad u \in D^{1,p}(\Omega). \qquad (3)$$

In the space $D^{1,p}(\Omega)$ we employ the following norm:

$$\|u\| = \|u\|_{D^{1,p}(\Omega)} := \left(\int\limits_{\Omega} \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx\right)^{1/p}, \quad \mu \in [0, \overline{\mu}).$$

By using the Hardy inequality (3) this norm is equivalent to the usual norm $\left(\int_{\Omega} |\nabla u|^p dx\right)^{1/p}$.
The operator $L := \left(|\nabla \cdot|^{p-2}\nabla \cdot - \mu\frac{|\cdot|^{p-2}}{|x|^p}\right)$ is positive in $D^{1,p}(\Omega)$ if $0 \le \mu < \overline{\mu}$.

Now, we define the space $W_k = D^{1,p}(\Omega) \times \ldots \times D^{1,p}(\Omega)$ with the norm

$$\big\|(u_1, \ldots, u_k)\big\|_k^p = \sum_{i=1}^{k} \|u_i\|^p.$$

Also, by Hardy inequality and Hardy–Sobolev inequality, for $0 \le \mu < \overline{\mu}$, $0 \le t < p$ and $p \le r \le p^*(t)$ we can define the best Hardy–Sobolev constant:

$$A_{\mu,t,r}(\Omega) = \inf_{u \in D^{1,p}(\Omega)\setminus\{0\}} \frac{\int_{\Omega}\left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx}{\left(\int_{\Omega}\frac{|u|^r}{|x|^t}dx\right)^{p/r}}. \tag{4}$$

In the important case when $r = p^*(t)$, we simply denote $A_{\mu,t,p^*(t)}$ as $A_{\mu,t}$. Note that $A_{\mu,0}$ is the best constant in the Sobolev inequality, namely,

$$A_{\mu,0}(\Omega) = \inf_{u \in D^{1,p}(\Omega)\setminus\{0\}} \frac{\int_{\Omega}\left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx}{\left(\int_{\Omega}|u|^{p^*}dx\right)^{p/p^*}}.$$

For any $0 \le \mu < \overline{\mu}$, by (2), (3), $0 \le t < p$ and the Minkowski's inequality, the following best constants are well defined:

$$S_{F,\mu} = S_{F,\mu}(\Omega) = \inf_{(u_1,\ldots,u_k)\in W_k\setminus\{(0,\ldots,0)\}} \frac{\sum_{i=1}^{k}\int_{\Omega}\left(|\nabla u_i|^p - \mu\frac{|u_i|^p}{|x|^p}\right)dx}{\left(\int_{\Omega}F(u_1,\ldots,u_k)dx\right)^{p/p^*}}. \tag{5}$$

Another important parameter is $A_{\mu,s,p}(\Omega)$, the (general) first eigenvalue of the operator L :

$$\lambda_1 = A_{\mu,s,p}(\Omega) = \inf_{u \in D^{1,p}(\Omega)\setminus\{0\}} \frac{\int_{\Omega}\left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx}{\int_{\Omega}\frac{|u|^p}{|x|^t}dx}. \tag{6}$$

Furthermore, λ_1 is positive and simple, the corresponding eigenfunction ϕ_1 does not change sign, the operator L admits a sequence of eigenvalues diverging to $+\infty$ [18, 19]. Without loss of generality, we can assume that $\phi_1 > 0$. Setting

$$E \triangleq \left\{ u \in D^{1,p}(\Omega); \int\limits_{\Omega} \frac{\phi_1^{p-1} u}{|x|^t} = 0 \right\}, \tag{7}$$

and

$$\lambda^* = \lambda^*_{\mu,s}(\Omega) = \inf_{u \in E \setminus \{0\}} \frac{\displaystyle\int_{\Omega} \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) dx}{\displaystyle\int_{\Omega} \frac{|u|^p}{|x|^t} dx}, \tag{8}$$

then we have $\lambda_1 < \lambda^*$ (see [15], Lemma 2.1).

Existence of nontrivial nonnegative solutions for elliptic equations with singular potentials were recently studied by several authors, but, essentially, only with a solely critical exponent. We refer, e. g., in bounded domains and for $p = 2$ to [6, 11, 12, 16], and for general $p > 1$ to [7, 8, 13 – 15] and the references therein. For example, Kang in [15] studied the following elliptic equation via the generalized Mountain – Pass theorem [17]:

$$-\Delta_p u - \mu \frac{|u|^{p-2} u}{|x|^p} = \frac{|u|^{p^*(t)-2} u}{|x|^t} + \lambda \frac{|u|^{p-2} u}{|x|^s}, \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p < N$, $0 \le s, t < p$ and $0 \le \mu < \overline{\mu} \triangleq \left(\frac{N-p}{p} \right)^p$. Also, the authors in [9] via the Mountain – Pass theorem of Ambrosetti and Rabinowitz [2], proved that

$$-\Delta_p u - \mu \frac{u^{p-1}}{|x|^p} = |u|^{p^*-1} + \frac{u^{p^*(s)-1}}{|x|^s} \quad \text{in} \quad \mathbb{R}^N$$

admits a positive solution in $\mathbb{R}^N$, whenever $\mu < \overline{\mu} \triangleq \left(\frac{N-p}{p} \right)^p$.

In this work, motivated by the above works we are interested to study the problem (1) by using the Mountain – Pass theorem of Ambrosetti and Rabinowitz [2]. We shall show that the system () has a positive weak solution.

This paper is divided into three sections, organized as follows. In Section 2, we establish some elementary results. In Section 3, we prove our main results (Theorems 2 and 3).

2. Local $(PS)_c$ condition. The corresponding energy functional of problem () is defined by

$$J(u) = \frac{1}{p} \|u\|^p_{W_k} - \frac{1}{p^*} \int\limits_{\Omega} F(u) dx - \frac{1}{p^*(t)} \sum_{i=1}^{k} \int\limits_{\Omega} \frac{|u_i|^{p^*(t)}}{|x|^t} dx - \frac{\lambda}{p} \sum_{i=1}^{k} \int\limits_{\Omega} \frac{|u_i|^p}{|x|^s} dx,$$

for each $u = (u_1, \ldots, u_k) \in W_k$. Then $J \in C^1(W_k, \mathbb{R})$.

Before proving the main results, we stat several lemmas.

The following lemma is well know, where we have employed the equivalent norm in $W^{1,p}(\Omega)$, see [13] for the case when $\mu = 0$.

Lemma 1. *Assume that* $0 \le s < p$, $p \le q \le p^*(s)$ *and* $0 \le \mu < \overline{\mu}$. *Then*:

(i) *there exists a constant* $C > 0$ *such that*

$$\left(\int\limits_{\Omega} \frac{|u|^q}{|x|^s}dx\right)^{1/q} \le C\|u\| \quad \textit{for all} \quad u \in D^{1,p}(\Omega),$$

(ii) *the map* $u \to \dfrac{u}{x^{s/q}}$ *from* $D^{1,p}(\Omega)$ *into* $L^q(\Omega)$ *is compact if* $p \le q < p^*(s)$.

Also, we need the following version of Brèzis – Lieb lemma [4].

Lemma 2. *Consider* $F \in C^1((\mathbb{R}^+)^k, \mathbb{R}^+)$ *with* $F(0,\ldots,0) = 0$ *and* $|F_{u_i}(u_1,\ldots,u_k)| \le \le C_1\left(\sum_{j=1}^n |u_j|^{p-1}\right)$ *for* $i = 1,\ldots,k$ *and some* $1 \le p < \infty$, $C_1 > 0$. *Let* $u_n = (u_1^n,\ldots,u_k^n)$ *be bounded sequence in* $L^p(\overline{\Omega},(\mathbb{R}^+)^k)$, *and such that* $u_n \rightharpoonup u = (u_1,\ldots,u_k)$ *weakly in* W_k. *Then one has*

$$\int\limits_{\Omega} F(u_n)dx \to \int\limits_{\Omega} F(u_n - u)dx + \int\limits_{\Omega} F(u)dx \quad \text{as} \quad n \to \infty.$$

Lemma 3. *Assume that* $0 \le s < p$ *and* $0 \le \mu < \overline{\mu}$. *Then the functional* J *satisfies the* $(PS)_c$ *condition for all*

$$0 < c < c^* := \min\left\{\frac{1}{N}S_{F,\mu}^{\frac{N}{p}}, \frac{p-t}{p(N-t)}(A_{\mu,t})^{(N-t)/(p-t)}\right\}. \tag{9}$$

Proof. Suppose $\{u_n = (u_1^n,\ldots,u_k^n)\} \subset W_k$ satisfies $J(u_n) \to c$ and $J'(u_n) \to 0$ with $c < c^*$. It is easy to show that $\{u_n\}$ is bounded in W_k and there exists $u = (u_1,\ldots,u_k)$ such that $u_n \rightharpoonup u$ up to a subsequence. Moreover, for $1 \le i \le k$, we may assume

$$u_i^n \rightharpoonup u_i \quad \text{weakly in} \quad D^{1,p}(\Omega),$$

$$u_i^n \rightharpoonup u_i \quad \text{weakly in} \quad L^{p^*(t)}(\Omega, |x|^t), \quad 0 < t \le p,$$

$$u_i^n \to u_i \quad \text{strongly in} \quad L^q(\Omega), \quad 1 \le q < p^*,$$

$$u_i^n \to u_i \quad \text{a.e. on} \quad \Omega.$$

Hence, we have $J'(u) = 0$ by the weak continuity of J. Let $\widetilde{u}_i^n = u_i^n - u_i$ for $1 \le i \le k$. Then we have

$$\int\limits_{\Omega} |\widetilde{u}_i^n|^p = o(1) \quad \text{for} \quad 1 \le i \le k \tag{10}$$

and by Brèzis – Lieb lemma [4], we obtain

$$\|\widetilde{u}_n\|_k^p \to \|u_n\|_k^p - \|u\|_k^p \quad \text{as} \quad n \to \infty, \tag{11}$$

$$\sum_{i=1}^k \int\limits_{\Omega} \frac{|\widetilde{u}_i^n|^{p^*(t)}}{|x|^t}dx = \sum_{i=1}^k \int\limits_{\Omega} \frac{|u_i^n|^{p^*(t)}}{|x|^t}dx - \sum_{i=1}^k \int\limits_{\Omega} \frac{|u_i|^{p^*(t)}}{|x|^t}dx + o(1), \tag{12}$$

and by Lemma 2,

$$\int\limits_{\Omega} F(\widetilde{u}_n)dx - \int\limits_{\Omega} F(u_n)dx \to \int\limits_{\Omega} F(u)dx \quad \text{as} \quad n \to \infty. \tag{13}$$

Since, $J(u_n) = c + o(1)$, $J'(u_n) = o(1)$ and (10) – (13), we can deduce that

$$\frac{1}{p}\|\widetilde{u}_n\|^p - \frac{1}{p^*}\int\limits_\Omega F(\widetilde{u}_n)dx - \frac{1}{p^*(t)}\sum_{i=1}^k \int\limits_\Omega \frac{|\widetilde{u}_i^n|^{p^*(t)}}{|x|^t}dx = c - J(u) + o(1),$$

and

$$\|\widetilde{u}_n\|^p - \int\limits_\Omega F(\widetilde{u}_n)dx - \sum_{i=1}^k \int\limits_\Omega \frac{|\widetilde{u}_i^n|^{p^*(t)}}{|x|^t}dx = o(1).$$

Now, we define

$$\alpha := \lim_{n\to\infty}\int\limits_\Omega F(\widetilde{u}_n)dx,$$

$$\beta := \lim_{n\to\infty}\sum_{i=1}^k \int\limits_\Omega \frac{|\widetilde{u}_i^n|^{p^*(t)}}{|x|^t}dx, \tag{14}$$

$$\gamma := \lim_{n\to\infty}\sum_{i=1}^k \int\limits_\Omega \left(|\nabla\widetilde{u}_i^n|^p - \mu\frac{|\widetilde{u}_i^n|^p}{|x|^p}\right)dx = \lim_{n\to\infty}\|\widetilde{u}_n\|^p.$$

Let $\xi \in C_0^\infty(\Omega)$ be such that $\xi|_\Omega \equiv 1$. Since $\xi u_n \in W_k$, and since $\lim_{n\to\infty}\langle J'(u_n), \xi u_n\rangle = 0$, using (10) – (13) and the definitions of α, β and γ in (14), we get that $\gamma \le \alpha + \beta$.

By (14), we obtain

$$\alpha^{p/p^*} = \lim_{n\to\infty}\left(\int\limits_\Omega F(\widetilde{u}_n)dx\right)^{p/p^*} \le \frac{1}{S_{F,\mu}}\lim_{n\to\infty}\|\widetilde{u}_n\|^p = \frac{1}{S_{F,\mu}}\gamma, \tag{15}$$

and by definition of $A_{\mu,t}$,

$$\beta^{p/p^*(t)} = \lim_{n\to\infty}\left(\sum_{i=1}^k \int\limits_\Omega \frac{|\widetilde{u}_i^n|^{p^*(t)}}{|x|^t}dx\right)^{p/p^*(t)} \le$$

$$\le \lim_{n\to\infty}\left[\left(\frac{1}{A_{\mu,t}}\right)^{p^*(t)/p}\sum_{i=1}^k\left(\int\limits_\Omega\left(|\nabla\widetilde{u}_i^n|^p - \mu\frac{|\widetilde{u}_i^n|^p}{|x|^p}\right)dx\right)^{p^*(t)/p}\right]^{p/p^*(t)} =$$

$$= \frac{1}{A_{\mu,t}}\lim_{n\to\infty}\left[\sum_{i=1}^k\|\widetilde{u}_i^n\|^{p^*(t)}\right]^{p/p^*(t)} =$$

$$= \frac{1}{A_{\mu,t}}\lim_{n\to\infty}\|\widetilde{u}_n\|^p = \frac{1}{A_{\mu,t}}\gamma. \tag{16}$$

From $\gamma \leq \alpha + \beta$, (15) and (16), we can obtain

$$\alpha^{p/p^*} \leq \frac{1}{S_{F,\mu}}\gamma \leq \frac{1}{S_{F,\mu}}\alpha + \frac{1}{S_{F,\mu}}\beta,$$
$$\alpha^{p/p^*}\left(1 - \frac{1}{S_{F,\mu}}\alpha^{\frac{p^*-p}{p^*}}\right) \leq \frac{1}{S_{F,\mu}}\beta. \tag{17}$$

On the other hand, $J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle = c + o(\|u_n\|) = c + o(1)$ as $n \to \infty$ since $(\|u_n\|)_{n\in\mathbb{N}}$ is bounded, which yields,

$$\left(\frac{1}{p} - \frac{1}{p^*}\right)\int\limits_\Omega F(u_n)dx + \left(\frac{1}{p} - \frac{1}{p^*(t)}\right)\sum_{i=1}^k \int\limits_\Omega \frac{|u_i^n|^{p^*(t)}}{|x|^t}dx = c + o(1) \tag{18}$$

as $n \to \infty$. Therefore

$$\int\limits_\Omega F(u_n)dx \leq cN + o(1) \tag{19}$$

as $n \to \infty$.

Moreover, by (19), we obtain

$$\alpha \leq cN. \tag{20}$$

Plugging (20) into (17), we have

$$\alpha^{p/p^*}\left(1 - \frac{1}{S_{F,\mu}}(cN)^{p/N}\right) \leq \frac{1}{S_{F,\mu}}\beta.$$

By the upper bound 9 on c there exists δ_1, depending on N, p, μ and c, such that $\alpha^{p/p^*} \leq \delta_1\beta$. Similarly, there exists δ_2, depending on N, p, μ, t and c, such that $\beta^{p/p^*(t)} \leq \delta_2\alpha$. In particular, it follows from these two latest inequalities that there exists $\epsilon_0 = \epsilon_0(N, p, \mu, s, c) > 0$ such that either

$$\alpha = \beta = 0 \quad \text{or} \quad \{\alpha \geq \epsilon_0 \ \text{ and } \ \beta \geq \epsilon_0\}. \tag{21}$$

Up to a subsequence, from (12) and (13) it follows that

$$c = J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle + o(1) \geq$$
$$\geq \frac{1}{N}\int\limits_\Omega F(u_n)dx + \frac{p-t}{p(N-t)}\sum_{i=1}^k \int\limits_\Omega \frac{|u_i^n|^{p^*(t)}}{|x|^t}dx + o(1) =$$
$$= \frac{1}{N}\left(\int\limits_\Omega F(u)dx + \alpha\right) + \frac{p-t}{p(N-t)}\left(\sum_{i=1}^k \int_\Omega \frac{|u_i|^{p^*(t)}}{|x|^t}dx + \beta\right)$$

as $n \to \infty$. From (21) and by assumption $c < c^*$ we get $\alpha = \beta = 0$. Up to a subsequence, $u_n \to u$ strongly in W_k.

Lemma 3 is proved.

Lemma 4 [15]. *Assume that* $1 < p < N$, $0 \le t < p$ *and* $0 \le \mu < \overline{\mu}$. *Then the limiting problem*

$$-\Delta_p u - \mu \frac{|u|^{p-1}}{|x|^p} = \frac{|u|^{p^*(t)-1}}{|x|^t} \quad in \quad \mathbb{R}^N \setminus \{0\},$$

$$u \in D^{1,p}(\mathbb{R}^N), \quad u > 0 \quad in \quad \mathbb{R}^N \setminus \{0\},$$

has positive radial ground states

$$V_\epsilon(x) \triangleq \epsilon^{(p-N)/p} U_{p,\mu}\left(\frac{x}{\epsilon}\right) = \epsilon^{(p-N)/p} U_{p,\mu}\left(\frac{|x|}{\epsilon}\right) \qquad \forall \epsilon > 0, \tag{22}$$

that satisfy

$$\int\limits_\Omega \left(|\nabla V_\epsilon(x)|^p - \mu \frac{|V_\epsilon(x)|^p}{|x|^p}\right) dx = \int\limits_\Omega \frac{|V_\epsilon(x)|^{p^*(t)}}{|x|^t} dx = (A_{\mu,t})^{(N-t)/(p-t)},$$

where $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ *is the unique radial solution of the limiting problem with*

$$U_{p,\mu}(1) = \left(\frac{(N-t)(\overline{\mu}-\mu)}{N-p}\right)^{1/(p^*(t)-p)}.$$

Furthermore, $U_{p,\mu}$ *have the following properties*:

$$\lim_{r\to 0} r^{a(\mu)} U_{p,\mu}(r) = C_1 > 0,$$

$$\lim_{r\to +\infty} r^{b(\mu)} U_{p,\mu}(r) = C_2 > 0,$$

$$\lim_{r\to 0} r^{a(\mu)+1} |U'_{p,\mu}(r)| = C_1 a(\mu) \ge 0,$$

$$\lim_{r\to +\infty} r^{b(\mu)+1} |U'_{p,\mu}(r)| = C_2 b(\mu) > 0,$$

where C_i, $i = 1, 2$, *are positive constants and* $a(\mu)$ *and* $b(\mu)$ *are zeros of the function*

$$f(\zeta) = (p-1)\zeta^p - (N-p)\zeta^{p-1} + \mu, \quad \zeta \ge 0, \quad 0 \le \mu < \overline{\mu},$$

that satisfy

$$0 \le a(\mu) < \frac{N-p}{p} < b(\mu) \le \frac{N-p}{p-1}.$$

Now, the p^*-homogeneity of F yields

$$F(u) \le M \left(\sum_{i=1}^k |u_i|^p\right)^{p^*/p} \qquad \text{for some constant} \qquad M > 0. \tag{23}$$

Recall that $p < p^*$ since $1 < p < N$.

Theorem 1. *Suppose* $0 \le s < p$ *and* $0 \le \mu < \overline{\mu}$. *Then*:

(i) $S_{F,\mu} = M^{-p/p^*} A_{\mu,0}$;

(ii) $S_{F,\mu}$ *has the minimizers* $(e_1 V_\epsilon(x), \ldots, e_k V_\epsilon(x))$ $\forall \epsilon > 0$, *where* $\Sigma_{i=1}^k e_i^p = 1$ *and* $V_\epsilon(x)$ *are the extremal functions of* $A_{\mu,0}$ *defined as in* (22) (*by plugging* $t = 0$ *in Lemma* 4).

Proof. (i) By (23) and the Minkowski's inequality

$$\left(\int\limits_{\Omega} F(u)dx\right)^{p/p^*} \le M^{p/p^*}\left(\int\limits_{\Omega}\left(\sum_{i=1}^{k}|u_i|^p\right)^{p^*/p} dx\right)^{p/p^*} \le$$

$$\le M^{p/p^*}\sum_{i=1}^{k}\left(\int\limits_{\Omega}|u_i|^{p^*}dx\right)^{p/p^*} \le$$

$$\le M^{p/p^*}A_{\mu,0}^{-1}\sum_{i=1}^{k}\|u_i\|^p = M^{p/p^*}A_{\mu,0}^{-1}\|u\|_{W_k}^p,$$

where $u=(u_1,\ldots,u_k)\in W_k$. So that

$$M^{p/p^*}\frac{\|u\|^p}{\left(\int_{\Omega}F(u)dx\right)^{p/p^*}} \ge A_{\mu,0}.$$

Consider now the map $u_0=e_0v_0$ where $e_0=(e_1,\ldots,e_k)\in(\mathbb{R}^+)^k$ satisfies $\sum_{i=1}^{k}e_i^p=1$ and $v_0\in D^{1,p}(\Omega)$ is an extremal function for $A_{\mu,0}$. Then

$$\left(\int\limits_{\Omega}F(u_0)dx\right)^{p/p^*} = M^{p/p^*}\left(\int\limits_{\Omega}|v_0|^{p^*}dx\right)^{p/p^*} =$$

$$= M^{p/p^*}A_{\mu,0}^{-1}\|v_0\|^p = M^{p/p^*}A_{\mu,0}^{-1}\|tv_0\|_{W_k}^p = M^{p/p^*}A_{\mu,0}^{-1}\|u_0\|_{W_k}^p.$$

So that

$$S_{F,\mu} = M^{-p/p^*}A_{\mu,0}. \tag{24}$$

(ii) From (5), (22) and (24) the desired result follows.

Theorem 1 is proved.

3. Main results. The main conclusions of this paper are summarized in the following theorems.

Theorem 2. *Assume that* $N + ps - s - p^2 > 0$, $\lambda \in (\lambda_1, \lambda^*)$ *and* $0 \le \mu \le \mu_1$, *where*

$$\mu_1 \triangleq \frac{N + ps - s - p^2}{p} \left(\frac{N - s}{p} \right)^{p-1}.$$

Then the problem () *has a positive solution.*

Theorem 3. *Assume that* $0 \le \mu < \overline{\mu}$, $\lambda > 0$ *and* $\lambda \in (\widetilde{\lambda}, \lambda_1)$, *where*

$$\widetilde{\lambda} = \lambda_1 \left(1 + \frac{M C^{-\frac{p^*}{p}} (N - p)}{N} \right) - c^{*p-t/N-t} \left[\frac{k(p-t)}{p(N-t)} \int\limits_{\Omega} |x|^{Nt+st-Ns-pt/p-t} \right]^{-p-t/N-t}.$$

Then the problem () *has a positive solution.*

In the following, we will give some estimates on the extremal function $V_\epsilon(x)$ defined in (22). For $m \in \mathbb{N}$ large, choose $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$, $0 \le \varphi(x) \le 1$, $\varphi(x) = 1$ for $|x| \le \dfrac{1}{2m}$, $\varphi(x) = 0$ for $|x| \ge \dfrac{1}{m}$, $\|\nabla\varphi(x)\|_{L^p(\Omega)} \le 4m$, set $u_\epsilon(x) = \varphi(x) V_\epsilon(x)$. For $\epsilon \to 0$, the behavior of u_ϵ has to be the same as that of V_ϵ, but we need precise estimates of the error terms. For $1 < p < N$, $0 \le s, t < p$ and $1 < q < p^*(s)$, we have the following estimates [15]:

$$\int\limits_{\Omega} \left(|\nabla u_\epsilon|^p - \mu \frac{|u_\epsilon|^p}{|x|^p} \right) dx = (A_{\mu,t})^{N-t/p-t} + O\big(\epsilon^{b(\mu)p+p-N}\big), \tag{25}$$

$$\int\limits_{\Omega} \frac{|u_\epsilon|^{p^*(t)}}{|x|^t} dx = (A_{\mu,t})^{N-t/p-t} + O\big(\epsilon^{b(\mu)p^*(t)-N+t}\big), \tag{26}$$

$$\int\limits_{\Omega} \frac{|u_\epsilon|^q}{|x|^s} dx \ge \begin{cases} C\epsilon^{N-s+\left(1-\frac{N}{p}\right)q}, & q > \dfrac{N-s}{b(\mu)}, \\ C\epsilon^{N-s+\left(1-\frac{N}{p}\right)q} |\ln\epsilon|, & q = \dfrac{N-s}{b(\mu)}, \\ C\epsilon^{q\left(b(\mu)+1-\frac{N}{p}\right)q}, & q < \dfrac{N-s}{b(\mu)}. \end{cases} \tag{27}$$

Lemma 5. *Assume that* $N + ps - s - p^2 > 0$, $\lambda \in (\lambda_1, \lambda^*)$, $\epsilon > 0$ *small enough and* $0 \le \mu \le \mu_1$, *where*

$$\mu_1 \triangleq \frac{N + ps - s - p^2}{p} \left(\frac{N - s}{p} \right)^{p-1}.$$

Then, there exist a function $u = (u_1, \ldots, u_k) \in W_k$ *such that*

$$\sup_{t \ge 0} J(tu) < c^* := \min \left\{ \frac{1}{N} S_{F,\mu}^{N/p}, \frac{p-t}{p(N-t)} (A_{\mu,t})^{N-t/p-t} \right\}.$$

Proof. We divide the proof into two steps.

***Step* 1.** We prove that under the assumptions of this lemma, there exists $(u_1, \ldots, u_k) \in W_k$ such that

$$\sup_{t\geq 0} J(tu_1,\ldots,tu_k) < \frac{1}{N}S_{F,\mu}.$$

We consider the functional $I\colon W\to\mathbb{R}$ defined by

$$I(u)=\frac{1}{p}\|u\|^p_{W_k}-\frac{1}{p^*}\int\limits_\Omega F(u)dx \quad \text{for all} \quad u\in W_k$$

and by (25), (26) and (27) in case $t=0$,

$$\int\limits_\Omega \left(|\nabla u_\epsilon|^p-\mu\frac{|u_\epsilon|^p}{|x|^p}\right)dx=(A_{\mu,0})^{N/p}+O\big(\epsilon^{b(\mu)p+p-N}\big),$$

$$\int\limits_\Omega |u_\epsilon|^{p^*}dx=(A_{\mu,0})^{N/p}+O\big(\epsilon^{b(\mu)p^*-N}\big),$$

$$\int\limits_\Omega \frac{|u_\epsilon|^q}{|x|^s}dx\geq\begin{cases} C\epsilon^{N-s+(1-N/p)q}, & q>\dfrac{N-s}{b(\mu)},\\ C\epsilon^{N-s+\big(1-N/p\big)q}\,|\ln\epsilon|, & q=\dfrac{N-s}{b(\mu)},\\ C\epsilon^{q\big(b(\mu)+1-N/p\big)q}, & q<\dfrac{N-s}{b(\mu)}.\end{cases}$$

Set $u_0=(e_1u_\epsilon,\ldots,e_ku_\epsilon)\in W_k$ where $(e_1,\ldots,e_k)\in(\mathbb{R}^+)^k$ and $\sum_{i=1}^k e_i^p=1$.

Also, we define the function $g_1(t):=J(te_1u_\epsilon,\ldots,te_ku_\epsilon)$, $t\geq 0$. Note that $\lim_{t\to+\infty}g_1(t)=-\infty$ and $g_1(t)>0$ as t is close to 0. Thus $\sup_{t\geq 0}g_1(t)$ is attained at some finite $t_\epsilon>0$ with $g_1'(t_\epsilon)=0$. Furthermore, $C'<t_\epsilon<C''$; where C' and C'' are the positive constants independent of ϵ.

Then, by the definition of $S_{F,\mu}$, we obtain

$$I(t_\epsilon e_1u_\epsilon,\ldots,t_\epsilon e_ku_\epsilon)\leq\frac{1}{N}\left(\frac{\left(\sum_{i=1}^k e_i^p\right)\int_\Omega\left(|\nabla u_\epsilon|^p-\mu\frac{|u_\epsilon|^p}{|x|^p}\right)dx}{\left(\int_\Omega F(e_1u_\epsilon,\ldots,e_ku_\epsilon)dx\right)^{p/p^*}}\right)^{N/p}\leq$$

$$\leq\frac{1}{N}\left(\frac{\int_\Omega\left(|\nabla u_\epsilon|^p-\mu\frac{|u_\epsilon|^p}{|x|^p}\right)dx}{M^{p/p^*}\left(\int_\Omega|u_\epsilon|^{p^*}dx\right)^{p/p^*}}\right)^{N/p}\leq$$

$$\leq\frac{1}{N}\left(\frac{1}{M^{p/p^*}}\right)^{N/p}\left(\frac{(A_{\mu,0})^{N/p}+O\big(\epsilon^{b(\mu)p+p-N}\big)}{(A_{\mu,0})^{N/p^*}+O\big(\epsilon^{\big(b(\mu)p^*-N\big)p/p^*}\big)}\right)^{N/p}\leq$$

$$\leq\frac{1}{N}\left(\frac{1}{M^{p/p^*}}\right)^{N/p}\left(\Big(A_{\mu,0}\Big)^{N/p}+O(\epsilon^{b(\mu)p+p-N})\right)=$$

$$= \frac{1}{N}\left(\frac{A_{\mu,0}}{M^{p/p^*}}\right)^{N/p} + O\big(\epsilon^{b(\mu)p+p-N}\big) = \frac{1}{N} S_{F,\mu}^{N/p} + O\big(\epsilon^{b(\mu)p+p-N}\big),$$

where the following fact has bas used:

$$\sup_{t\geq 0}\left(\frac{t^p}{p}A - \frac{t^{p^*}}{p^*}B\right) = \frac{1}{N}\left(\frac{A}{B^{\frac{p}{p^*}}}\right)^{N/p}, \quad A,\ B > 0.$$

Consequently,

$$J(t_\epsilon e_1 u_\epsilon, \ldots, t_\epsilon e_k u_\epsilon) \leq \frac{1}{N} S_{F,\mu}^{N/p} + O(\epsilon^{b(\mu)p+p-N}) - \frac{k\lambda m}{p}\int\limits_\Omega \frac{|t_\epsilon u_\epsilon|^p}{|x|^s}dx,$$

where $m := \min\{e_1^p, \ldots, e_n^p\}$.

If $p > \dfrac{N-s}{b(\mu)}$, from (27) we have

$$\int\limits_\Omega \frac{|t_\epsilon u_\epsilon|^p}{|x|^s}dx \geq C\epsilon^{N-s+p-N} = O\big(\epsilon^{N-s+p-N}\big).$$

Furthermore, $N - s + p - N < pb(\mu) + p - N$.

If $p = \dfrac{N-s}{b(\mu)}$, then $N - s + p - N = pb(\mu) + p - N$. From (27) we have

$$\int\limits_\Omega \frac{|t_\epsilon u_\epsilon|^p}{|x|^s}dx \geq C\epsilon^{N-s+p-N}|\ln\epsilon| = O\big(\epsilon^{N-s+p-N}|\ln\epsilon|\big).$$

Hence, if $\epsilon > 0$ small and $pb(\mu) - N + s \geq 0$, then we have

$$\sup_{t\geq 0} J(te_1u_\epsilon, \ldots, te_ku_\epsilon) < \frac{1}{N}S_{F,\mu}.$$

On the other hand, it is easy to verify that the function

$$f(\zeta) = (p-1)\zeta^p - (N-p)\zeta^{p-1} + \mu, \quad \zeta \geq 0,$$

has the only minimal point $\zeta'' = \dfrac{N-p}{p}$. Moreover, $f(\zeta)$ is decreasing in $(0, \zeta'')$ and is increasing in $(\zeta'', +\infty)$. Hence,

$$pb(\mu) - N + s \geq 0 \Longleftrightarrow b(\mu) \geq \frac{N-s}{p} \Longleftrightarrow$$

$$\Longleftrightarrow 0 = f(b(\mu)) \geq f\left(\frac{N-s}{p}\right) \Longleftrightarrow 0 \leq \mu \leq \mu_1$$

for $N + ps - s - p^2 > 0$.

***Step* 2.** We prove that under the assumptions of this lemma, there exists $(u_1, \ldots, u_k) \in W_k$ such that

$$\sup_{t\geq 0} J(tu_1, \ldots, tu_k) < c^*.$$

In case

$$\frac{1}{N}S_{F,\mu}^{N/p} \leq \frac{p-t}{p(N-t)}(A_{\mu,t})^{N-t/p-t},$$

we take $(u_1,\ldots,u_k) \in W_k \setminus \{(0,\ldots,0)\}$ as in step 1 to get the result. Otherwise we take $(u_1,\ldots,u_k) = (e_1u_\epsilon,\ldots,e_ku_\epsilon) \in \setminus\{(0,\ldots,0)\}$ where $(e_1,\ldots,e_k) \in (\mathbb{R}^+)^k$ and $\sum_{i=1}^k e_i^p = 1$ and u_ϵ satisfy (25)–(27).

Now, we using arguments similar to the first step, with I replace by:

$$\widetilde{I}(u) = \frac{1}{p}\|u\|_{W_k}^p - \frac{1}{p^*(t)}\sum_{i=1}^k \int\limits_\Omega \frac{|u_i|^{p^*(t)}}{|x|^t}dx \qquad \text{for all} \qquad u \in W_k.$$

Which gives the step 2.

Lemma 5 is proved.

***Proof of Theorem* 2.** Set $c = \inf_{h\in\Gamma}\max_{t\in[0,1]} J(h(t))$, where

$$\Gamma = \big\{h \in C([0,1],W_k) \mid h(0) = (0,0),\ J(h(1)) < 0\big\}.$$

For any $u = (u_1,\ldots,u_k) \in W_k \setminus \{(0,\ldots,0)\}$, from Lemma 1 (i) and the Minkowski's inequality, one can get

$$\int\limits_\Omega F(u)dx \leq M\left(\int\limits_\Omega \left(\sum_{i=1}^k |u_i|^p\right)^{p^*/p} dx\right)^{(p/p^*)\cdot(p^*/p)} \leq MC^{-(p^*/p)}\|u\|_{W_k}^p. \tag{28}$$

Then it follows that

$$J(u) = \frac{1}{p}\|u\|_{W_k}^p - \frac{1}{p^*}\int\limits_\Omega F(u)dx - \frac{1}{p^*(t)}\sum_{i=1}^k \int\limits_\Omega \frac{|u_i|^{p^*(t)}}{|x|^t}dx - \frac{\lambda}{p}\sum_{i=1}^k \int\limits_\Omega \frac{|u_i|^p}{|x|^s}dx \geq$$

$$\geq C\left(\|u\|_{W_k}^p - \|u\|_{W_k}^{p^*} - \|u\|_{W_k}^{p^*(t)}\right) \geq C\|u\|_{W_k}^p - C\|u\|_{W_k}^{p^*}.$$

Hence, there exists a constant $\rho > 0$ small such that

$$b := \inf_{\|u\|_{W_k}=\rho} J(u) > 0 = J(0,\ldots,0).$$

Since $J(tu) \to -\infty$ as $t \to \infty$, there exists $t_0 > 0$ such that $\|t_0u\| > \rho$ and $J(t_0u) < 0$. By the Mountain–Pass theorem [2], there exists a sequence $\{u_n\} \subset W_k$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. From Lemma 5 it follows that

$$0 < c \leq \sup_{t\in[0,1]} J(tt_0e_1u_\epsilon,\ldots,tt_0e_ku_\epsilon)) \leq \sup_{t\geq 0} J(te_1u_\epsilon,\ldots,te_ku_\epsilon)) < c^*.$$

By Lemma 3 there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \to u$ strongly in W_k. Thus we get a critical point $u = (u_1,\ldots,u_k)$ of J satisfying () and c is a critical value. Set $u^+ = \max\{u,0\}$. Replacing the terms

$$\int\limits_\Omega \frac{|u_i|^{p^*(t)}}{|x|^t}dx, \qquad \int\limits_\Omega F(u)dx, \qquad \int\limits_\Omega \frac{|u_|^p}{|x|^s}dx \qquad \text{for} \qquad 1 \leq i \leq k$$

in $J(u)$ by

$$\int\limits_\Omega \frac{|u_i^+|^{p^*(t)}}{|x|^t}dx, \qquad \int\limits_\Omega F(u^+)dx, \qquad \int\limits_\Omega \frac{|u_i^+|^p}{|x|^s}dx \qquad \text{for} \qquad 1 \le i \le k$$

respectively and repeating the above process, we get a nonnegative solution $u = (u_1, \ldots, u_k)$ to (). Also, by the maximum principle we deduce that $u_i > 0$ in Ω for $1 \le i \le k$.

Theorem 2 is proved.

***Proof of Theorem* 3.** The proof follows the same lines as that in [3]. Let $w = u_1 = \cdots = u_k = = \tau\phi_1$, $\tau > 0$. Then by (28) and the Hölder inequality we obtain

$$J(u_1, \ldots, u_k) = \frac{\lambda_1 - \lambda}{p} \sum_{i=1}^k \int\limits_\Omega \frac{|w|^p}{|x|^s}dx - \frac{1}{p^*}\int\limits_\Omega F(w, \ldots, w)dx - \frac{1}{p^*(t)}\sum_{i=1}^k \int\limits_\Omega \frac{|w|^{p^*(t)}}{|x|^t}dx \le$$

$$\le \frac{\lambda_1 - \lambda}{p} \sum_{i=1}^k \int\limits_\Omega \frac{|w|^p}{|x|^s}dx + \frac{\lambda_1 M C^{-(p^*/p)}}{p^*}\sum_{i=1}^k \int\limits_\Omega \frac{|w|^p}{|x|^s}dx - \frac{1}{p^*(t)}\sum_{i=1}^k \int\limits_\Omega \frac{|w|^{p^*(t)}}{|x|^t}dx =$$

$$= \frac{k}{p}\left(\lambda_1\left(1 + \frac{MC^{-(p^*/p)}(N-p)}{N}\right) - \lambda\right)\int\limits_\Omega \frac{|w|^p}{|x|^s}dx - \frac{k}{p^*(t)}\int\limits_\Omega \frac{|w|^{p^*(t)}}{|x|^t}dx \le$$

$$\le \frac{k}{p}\left(\lambda_1\big(1 + \frac{MC^{-p^*/p}(N-p)}{N}\big) - \lambda\right)\left(\int\limits_\Omega \frac{|w|^{p^*(t)}}{|x|^t}dx\right)^{p/(p^*(t))} \times$$

$$\times \left(\int\limits_\Omega |x|^{(Nt+st-Ns-pt)/(p-t)}dx\right)^{(p-t)/(N-t)} - \frac{k}{p^*(t)}\int\limits_\Omega \frac{|w|^{p^*(t)}}{|x|^t}dx \le$$

$$\le \left(\lambda_1\big(1 + \frac{MC^{-\frac{p^*}{p}}(N-p)}{N}\big) - \lambda\right)^{(N-t)/(p-t)} \frac{k(p-t)}{p(N-t)}\int\limits_\Omega |x|^{(Nt+st-Ns-pt)/(p-t)}dx,$$

where we have used the fact that

$$\max_{\tau \ge 0}\left(c_1\tau^p - c_2\tau^{p^*(t)}\right) = \frac{c_1(p-t)}{N-t}\left(\frac{c_1(N-p)}{c_2(N-t)}\right)^{(N-p)/(p-t)} \qquad \forall\, c_1,\ c_2 > 0.$$

If $\lambda \in (\widetilde{\lambda}, \lambda_1)$, then

$$\max_{\tau \ge 0} J(\tau\phi_1, \ldots, \tau\phi_1) \le c^*.$$

Hence, we can obtain a PS-sequence in the cone of nonnegative functions, which has a weak limit $(u_1, \ldots, u_k)$ with $u_i \ge 0$ and $u_i \ne 0$ for $1 \le i \le k$. By the maximum principle [20], we obtain that $u_i > 0$ in Ω and $(u_1, \ldots, u_k)$ is a positive solution of ().

Theorem 3 is proved.

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