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S. Varbanets (Odessa Nat. Univ)

GENERALIZED TWISTED KLOOSTERMAN SUM OVER $\mathbb{Z}[i]$

УЗАГАЛЬНЕНА ГІБРИДНА СУМА КЛОСТЕРМАНА НАД $\mathbb{Z}[i]$

The twisted Kloosterman sums over $\mathbb{Z}$ were studied by V. Bykovsky, A. Vinogradov, N. Kuznetsov, R. W. Bruggeman, R. J. Miatello, I. Pacharoni, A. Knightly, and C. Li. In our paper, we obtain similar estimates of $K_\chi(\alpha,\beta;\gamma;q)$ over $\mathbb{Z}[i]$ and improve estimates obtained for the sums of this kind with Dirichlet character $\chi \pmod{q_1}$, where $q_1 \mid q$.

Узагальнені суми Клостермана з характером над $\mathbb{Z}$ вивчали В. Биковський, А. Виноградов, М. Кузнєцов, А. Найтлі та С. Лі. У статті отримано аналогічні оцінки для $K_\chi(\alpha,\beta;\gamma;q)$ над $\mathbb{Z}[i]$, а також уточнено оцінки таких сум з характером Діріхле $\chi \pmod{q_1}$, де $q_1 \mid q$.

1. Introduction. The classic Kloosterman sums appeared first in the work of Kloosterman [8] in connection with the representation of natural numbers by binary quadratic forms. The Kloosterman sum is an exponential sum over a reduced residue system modulo q:

$$K(a,b;q) := \sum_{\substack{x=1\\(x,q)=1}}^{q} e^{2\pi i\frac{ax+bx^{-1}}{q}} \quad (a,b\in\mathbb{Z} \quad q>1 \text{ is a positive integer})$$

here and in the seque x^{-1} denote the reciprocal to x modulo q, i.e., $xx^{-1}\equiv 1 \pmod q$.

By the relation for $q=q_1q_2$, $(q_1,q_2)=1$,

$$K(a,b;q) = K(aq_2',bq_2';q_1)\cdot K(aq_1',bq_1';q_2)$$

follows that suffices to obtain the estimations $K(a,b;q)$ only for a case $q=p^n$, p be a prime, $n\in\mathbb{N}$.

The greatest difficultly in an estimation of the Kloosterman sums provides the case $q=p$. The estimation $K(a,b;p)\ll p^{3/4}$ under a condition $(a,b,p)=1$ was obtained in the named work of Kloosterman, and then Davenport [5] improved on it up to $\ll p^{2/3}$. A. Weil [14] proved the Riemann hypothesis for algebraic curves of over finite field and obtained for $K(a,b;p)$ the best possible estimation $\ll p^{1/2}$.

Davenport [5] studied the general Kloosterman sums over finite field with the multiplicative character χ of this field

$$K_\chi(a,b;p) = \sum_{x\in\mathbb{F}_p^*} \chi(x) e^{2\pi i\frac{ax+bx^{-1}}{p}}.$$

The sums containing simultaneously multiplicative and additive characters call twisted or hybrid sums.

The further generalizations of the Kloosterman sums concerned with a substitution of a prime field $\mathbb{F}_p$ on it a finite expansion $\mathbb{F}_q$, $q=p^n$, $1<n\in\mathbb{N}$. The generalizations of the Kloosterman sums concerned with theory of modular forms studied in the works Kuznetsov [10, 11], Bruggeman [2], Deshoiller and Iwaniec [6], Proskurin [12], R. W. Bruggeman, R. J. Miatello, I. Pacharoni [1], A. Knightly, and C. Li [9].

Let consider the ring of the Gaussian integers $\mathbb{Z}[i]$. For Gaussian integers α, β, γ we can define the Kloosterman sum

$$K(\alpha,\beta;\gamma)=\sum_{\substack{x\in\mathbb{Z}[i]\\ x\,(\mathrm{mod}\,\gamma)\\ (x,\gamma)=1}}\exp\left(\pi i\,\mathrm{Sp}\frac{\alpha x+\beta x^{-1}}{\gamma}\right).$$

R. W. Bruggeman and Y. Motohashi [3] obtained the estimation

$$K(\alpha,\beta;\gamma)\ll 2^{\nu(\gamma)}N(\gamma)^{1/2}N((\alpha,\beta,\gamma))^{1/2},$$

where $\nu(\gamma)$ is the number distinct prime divisors of γ; (α,β,γ) denotes the greatest common divisor of α, β, γ.

In [13] we considered two type of generalized Kloosterman sums over $\mathbb{Z}[i]$

$$K_\chi(\alpha,\beta;k;\gamma)=\sum_{\substack{x\,(\mathrm{mod}\,\gamma)\\ (x,\gamma)=1}}\chi(x)\exp\left(\pi i\,\mathrm{Sp}\frac{\alpha x^k+\beta x^{-1^k}}{\gamma}\right),$$

where α, β, $\gamma\in\mathbb{Z}[i]$, χ is multiplicative character modulo γ, and

$$\widetilde{K}(\alpha,\beta;h,q;k)=\sum_{\substack{x,y\in\mathbb{Z}[i]\\ x,y\,(\mathrm{mod}\,\gamma)\\ N(xy)\equiv h\,(\mathrm{mod}\,q)}}e_q\left(\frac{1}{2}\,\mathrm{Sp}(\alpha x^k+\beta y^k)\right),$$

where $\alpha,\ \beta\in\mathbb{Z}[i],\ h,q\in\mathbb{N},\ (h,q)=1$.

We call $K(\alpha,\beta;k;\gamma,\chi)$ the twisted power Kloosterman sum and $\widetilde{K}(\alpha,\beta;h,q;k)$ call the norm Kloosterman sum.

In this paper we obtain the estimations of generalized Kloosterman sum with a Dirichlet character over the ring of the Gaussian integers which extend the A. Knightly, C. Li [9] results.

***Remark* 1.1.** We denote $G:=\mathbb{Z}[i]$ the ring of the Gaussian integers

$$G=\left\{a+bi\middle|a,b,\in\mathbb{Z},i^2=-1\right\}.$$

For the designation of the Gaussian integers we shall use the Greek letters α, β, γ, ξ, η; a Gaussian prime number denote through $\mathfrak{p}$ if $\mathfrak{p}\notin\mathbb{Z}$. For $\alpha\in\mathbb{Z}[i]$ we put $\mathrm{Sp}(\alpha)=\alpha+\overline{\alpha}=2\Re(\alpha),\ N(\alpha)=$ $=\alpha\cdot\overline{\alpha}$, where $\overline{\alpha}$ denotes a complex conjugate with α; $\mathrm{Sp}(\alpha)$ and $N(\alpha)$ we name a trace and a norm (respectively) of α from $\mathbb{Q}(i)$ into $\mathbb{Q}$.

The writing $a\in\mathbb{Z}_q$ (respectively, $\alpha\in G_\gamma$) under the sign Σ denotes that $a\in\mathbb{Z}$ (respectively, $\alpha\in G$) and a (respectively, α) runs a complete residue system modulo q (modulo γ). Analogous, $a\in\mathbb{Z}_q^*$ (respectively, $\alpha\in G_\gamma^*$) denotes $a\in\mathbb{Z}$ (respectively, $\alpha\in G$) and runs a reduced residue system modulo q (respectively, modulo γ).

The writing $\sum_{(U)}$ denotes that the summation runs over the region U which describes separately. For $A\in\mathbb{N}$ (or $\alpha\in G$) put $\nu_p(A)=a$ (or $\nu_{\mathfrak{p}(\alpha)}=a$) if $p^a\|A$ (or $\mathfrak{p}^a\|\alpha$). Moreover, $\exp(z)=e^z$, $e_q(z)=e^{2\pi i\frac{z}{q}}$ for $q\in\mathbb{N}$; the Vinogradov symbol as in $f(x)\ll g(x)$ means that $f(x)=O(g(x))$.

2. Auxiliary results. For the proof of our main results the following lemmas are needed.

Lemma 2.1. *Let* $f(x) \in \mathbb{Z}[x]$, $f(x) = a_1x + a_2x^2 + p^{\lambda_3}a_3x^3 + \ldots + p^{\lambda_k}a_kx^k$, $\lambda_j > 0$, $j = 3, \ldots, k$; $(a_i, p) = 1$, $i = 2, 3, \ldots, k$; $p > 2$ *be a prime number. Then for* $m \in \mathbb{N}$ *we have*

$$\sum_{x\in\mathbb{Z}_{p^m}} e_{p^m}(f(x)) = \varepsilon(m)p^{m/2}e_{p^m}(F(a_1, \ldots, a_k)), \tag{2.1}$$

where $F(x_1, \ldots, x_k) \in \mathbb{Z}[x_1, \ldots, x_k]$, *and, moreover,*

$$F(a_1, a_2, \ldots, a_k) \equiv -a_1^2(2a_2)^{-1} \pmod p,$$

$$\varepsilon(m) = \begin{cases} 1 & if \quad m \quad \text{is even,} \\ i^{\left(\frac{p-1}{2}\right)^2} & if \quad m \quad \text{is odd.} \end{cases}$$

Proof. Setting $x = y + p^{m-1}z$, $y \in \{0, 1, \ldots, p^{m-1} - 1\}$, $z \in \{0, 1, \ldots, p - 1\}$, we obtain

$$S := \sum_{x\in\mathbb{Z}_{p^m}} e_{p^m}(f(x)) = \sum_{y\in\mathbb{Z}_{p^{m-1}}} \sum_{z\in\mathbb{Z}_p} e_{p^m}(f(y) + p^{m-1}zf'(y)).$$

The sum over z gives zero if $f'(y) \not\equiv 0 \pmod p$.

We have $f'(y) = a_1 + 2a_2y \pmod p$. Thus

$$S = e_{p^m}(f(y_0))p \sum_{y\in\mathbb{Z}_{p^{m-1}}} e_{p^{m-2}}(g(y)),$$

where $y_0 \in \mathbb{Z}_p$, $a_1 + 2a_2y_0 \equiv 0 \pmod p$, $g(y) = \dfrac{f(y_0 + py) - f(y_0)}{p^2} = b_1y + b_2y^2 + p^{\mu_3}b_3y^3 + \ldots$ $\ldots + p^{\mu_k}b_ky^k$, $b_1 \equiv \dfrac{a_1 + 2a_2y_0}{p} \pmod p$, $b_2 \equiv a_2 \pmod p$, $b_j \equiv a_j \pmod p$, $\mu_j > 1$.

These considerations we continue further.

Thereby for $m \equiv 0 \pmod 2$ we obtain

$$S = p^{m/2}e_{p^m}\big(f(y_0) + p^2g(y_1) + \ldots\big). \tag{2.2}$$

For n is odd we have

$$S = p^{\frac{m-1}{2}}e_{p^m}\big(f(y_0) + p^2g(y_1 + \ldots)\big)\sum_{x\in\mathbb{Z}_p} e_p(b_1x + a_2x^2) =$$

$$= p^{\frac{m}{2}}i^{\left(\frac{p-1}{2}\right)^2}e_{p^m}\big(f(y_0) + p^2g(y_1) + \ldots + p^{m-1}b_1'\big). \tag{2.3}$$

Take into account that $f(y_0) \equiv -a_1^2(2a_2)^{-1} \pmod p$, we prove Lemma 2.1.

Lemma 2.2. *Let* $\mathfrak{p}$ *be the Gaussian prime number,* $\alpha_1, \ldots, \alpha_k \in G$, $(\alpha_j, \mathfrak{p}) = 1$, $j = 2, 3, \ldots;$ λ_j *be a positive integer,* $j = 3, \ldots, k$. *Then the relations*

$$\sum_{\xi\in G_{\mathfrak{p}}^m}\exp\left(\pi i\,\mathrm{Sp}\left(\frac{\alpha_1\xi+\alpha_2\mathfrak{p}\xi^2+\alpha_3\mathfrak{p}^{\lambda_3}\xi^3+\ldots+\alpha_k\mathfrak{p}^{\lambda_k}\xi^k}{\mathfrak{p}^m}\right)\right)=$$

$$=\begin{cases}0 & \text{if}\quad \alpha_1\not\equiv 0 \pmod{\mathfrak{p}},\quad \mathfrak{p}\neq 1+i,\\ e^{\pi i\,\mathrm{Sp}\left(\frac{F_1(\alpha_1,\ldots,\alpha_k)}{\mathfrak{p}^m}\right)}N(\mathfrak{p})^{\frac{m+1}{2}} & \text{if}\quad \alpha_1\equiv 0 \pmod{\mathfrak{p}},\quad \mathfrak{p}\neq 1+i,\\ 0 & \text{if}\quad \alpha_1\not\equiv 0 \pmod{\mathfrak{p}^2},\quad \mathfrak{p}=1+i,\\ e^{\pi i\,\mathrm{Sp}\left(\frac{F_2(\alpha_1,\ldots,\alpha_k)}{\mathfrak{p}^m}\right)} & \text{if}\quad \alpha_1\equiv 0 \pmod{\mathfrak{p}^2},\quad \mathfrak{p}=1+i,\end{cases} \tag{2.4}$$

hold, where the polynomials F_1, F_2 *are similar to* F *from Lemma* 2.1.

This assertion can be proved exactly in the same way as Lemma 2.1.

Lemma 2.3. *Let* $p>2$ *be a prime number,* $h,m\in\mathbb{N}$, $m>1$, $(h,p)=1$. *Then for any* $\alpha_1,\alpha_2\in G$ *the estimate*

$$\left|\sum_{\xi\in G_{p^m}}e^{\pi i\,\mathrm{Sp}\left(\frac{\alpha_1\xi+\alpha_2\xi^2+phN(\xi)+p^2\alpha_3\xi^3+\ldots+p^2\alpha_k\xi^k}{p^m}\right)}\right|\le p^{m+1} \tag{2.5}$$

holds.

Proof. Let $\alpha_1=a_1+ib_1$, $\alpha_2=a_2+ib_2$, $\xi=x+iy$. Then

$$\mathrm{Sp}(\alpha_1\xi)=2(a_1x-b_1y),\qquad \mathrm{Sp}(\alpha_2\xi^2)=2(a_2x^2-a_2y^2-2b_2xy).$$

Hence,

$$S:=\sum_{\xi\in G_{p^m}}e^{\pi i\,\mathrm{Sp}\left(\frac{\alpha_1\xi+\alpha_2\xi^2+phN(\xi)+p^2\alpha_3\xi^3+\ldots}{p^m}\right)}=$$

$$=\sum_{x,y\in\mathbb{Z}_{p^m}}e^{2\pi i\frac{a_1x-b_1y+pa_2x^2-pa_2y^2-2pb_2xy+ph(x^2+y^2)+p^2f(x,y)}{p^m}}, \tag{2.6}$$

where $f(x,y)$ is a polynomial without free term.

We have that $(a_2+h,p)=1$ or $(a_2-h,p)=1$. Let $(a_2+h,p)=1$. We can write

$$S=\sum_{y\in\mathbb{Z}_{p^m}}e^{-2\pi i\frac{b_1y-p(h+a^2)y^2}{p^m}}\sum_{x\in\mathbb{Z}_{p^m}}e^{2\pi i\frac{(a_1-2pb_2y)x+p(a_2h)x^2+p^2f(x,y)}{p^m}}.$$

It is well-known that the summation on x (or, respectively, y) gives zero if $a_1\not\equiv 0 \pmod{p}$ or $b_1\not\equiv 0 \pmod{p}$.

Thus we will set that $\alpha_1=p(a_1+ib_1)$. Then we obtain

$$S=\sum_{y\in\mathbb{Z}_{p^m}}e^{-2\pi i\frac{b_1y+(h+b_2)y^2}{p^{m-1}}}\sum_{x\in\mathbb{Z}_{p^m}}e^{2\pi i\frac{(a_1-b_2y)x+(a_2+h)x^2+pf(x,y)}{p^{m-1}}}=$$

$$= p^2 \sum_{y \in \mathbb{Z}_{p^{m-1}}} e^{-2\pi i \frac{b_1 y + (h+b_2)y^2}{p^{m-1}}} \sum_{x \in \mathbb{Z}_{p^{m-1}}} e^{2\pi i \frac{(a_1 - b_2 y)x + (a_2+h)x^2 + pf(x,y)}{p^{m-1}}} :=$$

$$:= \sum_{y \in \mathbb{Z}_{p^{m-1}}} e^{-2\pi i \frac{b_1 y + (h+b_2)y^2}{p^{m-1}}} \cdot S_1(y), \tag{2.7}$$

say.

The sum $S_1(y)$ we can calculate by Lemma 2.1:

$$S_1(y) = \varepsilon(m-1) p^{\frac{m-1}{2}} e_{p^{m-1}}(f(y_0) + p^2 g(y_1) + \ldots), \tag{2.8}$$

where $f(y_0) \equiv \dfrac{(a_1 - 2pb_2 y)^2}{2(a_2+h)} \equiv \dfrac{a_1^2}{2(a_2+h)} - \dfrac{2phb_2}{a_2+h} y + \dfrac{2p^2 b_2^2}{a_2+h} y^2 \pmod{p}$.

Thereby from (2.7), (2.8) by Lemma 2.1 we infer

$$\left| \sum_{\xi \in G_{p^m}} e^{\pi i \operatorname{Sp}\left(\frac{\alpha_1 \xi + \alpha_2 \xi^2 + phN(\xi) + p^2 \alpha_3 \xi^3 + \ldots + p^2 \alpha_k \xi^k}{p^m}\right)} \right| \le p^{m+1}.$$

Lemma 2.3 is proved.

3. Preliminary result. Let a modulus $q_1 \in \mathbb{Z}^+$, and let χ be a Dirichlet character modulo q_1. Over the ring of Gaussian integers $G = \mathbb{Z}[i]$ we define the following generalized twisted Kloosterman sum with the multiplicative function χ for any q, $q \equiv 0 \pmod{q_1}$:

$$K_\chi(\alpha, \beta; \gamma; q) = \sum_{x, y \in G_q} \overline{\chi(N(x))} e^{2\pi i \operatorname{Sp}\left(\frac{\alpha x + \beta y}{q}\right)}. \tag{3.1}$$

Note that χ is not generally a Dirichlet character modulo q, because it can be happened that $\chi(N(x)) \neq 0$ when $(x, q) \neq 1$.

In the special case where $\gamma = 1$ and $q_1 = q$ we obtain the twisted Kloosterman sum with a character χ defined by

$$K_\chi(\alpha, \beta; q) = \sum_{\substack{x, y \in G_q^* \\ xy \equiv 1 \pmod{q}}} \overline{\chi(x)} e^{\pi i \operatorname{Sp}\left(\frac{\alpha x + \beta y}{q}\right)}. \tag{3.2}$$

The generalized twisted Kloosterman sum $K_\chi(\alpha, \beta; q)$ has the property of quasimultiplicativity at q, i.e., for $q = q'q''$, $(q', q'') = 1$ the equality

$$K_\chi(\alpha, \beta; \gamma; q) = K_{\chi_1}(\alpha, \beta_1; \gamma; q') \cdot K_{\chi_2}(\alpha, \beta_2; \gamma; q'') \tag{3.3}$$

holds, where χ_1, χ_2 are characters inducted by character χ, and β_1, β_2 define from the congruence

$$\beta = \beta_1 (q'')^2 + \beta_2 (q')^2 \pmod{q}.$$

Thus in the sequence it will regard only the Kloosterman sum $K_\chi(\alpha, \beta; \gamma; q)$ with $q = p^m$, $p > 2$ be a prime number from $\mathbb{Z}$.

Lemma 3.1. *Let $p > 2$ be a prime number. Suppose $q = p^m$ and χ is a Dirichlet character of conductor p^{m_0}, $m_0 \le m$. Then for any Gaussian integers α, β*

$$|K_\chi(\alpha,\beta;p^m)| := \left| \sum_{x,y \in G^*_{p^m}} \chi_{p^m}(N(x)) e_{p^m}(\Re(\alpha x + \beta x^{-1})) \right| \le \varepsilon_p N(p)^{m/2}, \tag{3.4}$$

where $\varepsilon_p = \begin{cases} 2 & \text{if } p \equiv 3 \pmod 4, \\ 4 & \text{if } p \equiv 1 \pmod 4. \end{cases}$

Proof. In the case $m = 1$ we obtained the required result for principal χ using the Weil's result on the estimate of exponential sum on an algebraic curve over finite field, and extended to nonprincipal character χ (see [13]).

Now we consider the case $m > 1$. Without loss generality we suppose that $(\alpha, \beta, p) = 1$.

Since $xy \equiv 1 \pmod q$, $q = p^m$, we write $y = x^{-1}$. Then putting $x = \xi + p^{m-1}$ we have

$$K_\chi(\alpha,\beta;p^m) = \sum_{x \in G^*_{p^m}} \chi_{p^m}(N(x)) e_{p^m}(\Re(\alpha x + \beta x^{-1})) =$$

$$= \sum_{\xi \in G^*_{p^{m-1}}} \sum_{\eta \in G_p} \chi_{p^m}(N(\xi + p^{m-1}\eta)) e_{p^m}(\Re(\alpha\xi + \beta\xi^{-1})) e_p(\Re((\alpha - \beta\xi^{-2})\eta)) =$$

$$= \sum_{\xi \in G^*_{p^{m-1}}} \chi_{p^m}(N(\xi)) e_{p^m}(\Re(\alpha\xi + \beta\xi^{-1})) \times$$

$$\times \sum_{\eta \in G_p} \chi_{p^m}(N(1 + p^{m-1}\xi^{-1}\eta)) e_p(\Re((\alpha - \beta\xi^{-1})\eta)). \tag{3.5}$$

Let $\chi_{p^m}(A)$ is defined by the relation

$$\chi_{p^m}(A) = \begin{cases} e^{2\pi i \frac{\nu \operatorname{ind} A}{p^{m-1}(p-1)}} & \text{if} \quad (A,p) = 1, \\ 0 & \text{if} \quad p \mid A, \end{cases}$$

where $\nu \in \{0, 1, \ldots, p^{m-1}(p-1) - 1\}$, $\operatorname{ind} A$ denotes the index of integer A, $(A,p) = 1$, relatively to the fixed primitive root modulo p^n in $\mathbb{Z}$.

Take into account that $\operatorname{ind}(1 + p^\ell B) = p^{m-\ell-1}(p-1)u^{-1}B$ with a some u, $(u,p) = 1$, u is not depend on B, if $(B,p) = 1$, and $N(1 + p^{m-1}\xi^{-1}\eta) \equiv 1 + p^{m-1} 2\Re(\xi^{-1}\eta) = 1 + p^{m-1} \operatorname{Sp}(\xi^{-1}\eta)$, we infer

$$\chi_{p^m}(N(1 + p^{m-1}\xi^{-1}\eta)) = e_{p^m}(\nu \Re(\xi^{-1}\eta)). \tag{3.6}$$

Now from (3.5), (3.6) it follows

$$K_\chi(\alpha,\beta;p^m) = \sum_{\xi \in G^*_{p^m{}_{-1}}} \chi_{p^m}(N(\xi)) e_{p^m}(\Re(\alpha\xi + \beta\xi^{-1})) \sum_{\eta \in G_p} e_p\left(\Re\left(\frac{(\nu\xi^{-1} + \alpha - \beta\xi^{-2})\eta}{p}\right)\right). \tag{3.7}$$

Let $Y(\alpha, \beta, \nu)$ is the set of solutions of the congruence

$$\alpha u^2 + \nu u - \beta \equiv 0 \pmod p), \; u \in G^*_{p^m}.$$

It is clear that $|Y(\alpha, \beta, \nu)| \le \begin{cases} 2 & \text{if } p \equiv 3 \pmod 4, \\ 4 & \text{if } p \equiv 1 \pmod 4. \end{cases}$

From (3.7) we have

$$K_\chi(\alpha, \beta; p^m) = p^2 \sum_{\xi_0 \in Y} \chi_{p^m}(N(\xi_0)) \times$$

$$\times \sum_{\xi \in G_{p^{m-2}}} \chi_{p^m}(N(1 + p\xi_0^{-1}\xi)) e_{p^m}(\Re(\alpha(\xi_0 + p\xi) + \beta\xi_0^{-1} - \beta\xi_0^{-2}p\xi + \ldots)) =$$

$$= p^2 \sum_{\xi_0 \in Y} \chi_{p^m}(N(\xi_0)) e_{p^m}(\Re(\alpha\xi_0 + \beta\xi_0^{-1})) \times$$

$$\times \sum_{\xi \in G_{p^{m-2}}} e_{p^{m-2}} \left(\Re \left(\frac{(\nu\xi_0^{-1} + \alpha - \beta\xi_0^{-2})}{p} \xi + \beta\xi_0^{-3}\xi^2 + p\beta\xi_0^{-1}\xi^3 + \ldots \right) \right). \tag{3.8}$$

Now, Lemma 2.2 gives

$$|K_\chi(\alpha, \beta; p^m)| \le \varepsilon_p N(p)^{m/2},$$

where $\varepsilon_p = \begin{cases} 2 & \text{if } p \equiv 3 \pmod 4, \\ 4 & \text{if } p \equiv 1 \pmod 4. \end{cases}$

Lemma 3.1 is proved.

Now, by (3.3) we infer immediately the next corollary.

***Corollary* 3.1.** *For any $\alpha, \beta \in G$ and every Dirichlet character χ modulo q, $q \in \mathbb{N}$ we have*

$$|K_\chi(\alpha, \beta; q)| \le \overline{\tau}(q) \sqrt{N((\alpha, \beta, q))} N(q)^{1/2},$$

where $\overline{\tau}(q)$ denotes the number of divisors q over G:

$$\overline{\tau}(q) = \sum_{\delta | q}{}^{*} 1,$$

(*here $*$ denotes that δ runs all nonassociated divisors of q over G*).

4. Main results. Now we will investigate the generalized twisted Kloosterman sum $K_\chi(\alpha, \beta; \gamma, q)$ with parameters $\alpha, \beta, \gamma \in G$, χ be a Dirichlet character modulo q_1, $q_1 \mid q$. We put $q = q'q''$, where $(q', q'') = 1$, q' consists from the same prime numbers as q_1, and hence, $q_1 | q'$.

From

$$G_q = G_{q'} \times G_{q''}$$

we deduce that χ can considers as a multiplicative function on G_q and it has a canonical factorization $\chi = \chi_{q'} \cdot \chi_{q''}$ on G_q, where $\chi_{q'}$ is the Dirichlet character modulo q', viewed as a function on G_q, and $\chi_{q''}$ is the constant function 1 on G_q.

Thus we have

$$K_\chi(\alpha, \beta; \gamma; q'q'') = K_{\chi_{q'}}(\alpha_1, \beta_1; \gamma; q') \cdot K_{\chi_{q''}}(\alpha_2, \beta_2; \gamma; q''). \tag{4.1}$$

Hence, for $q = \prod_{p|q} p^{a_p}$, $q_1 = \prod_{p|q} p^{b_p}$ we deduce

$$K_\chi(\alpha, \beta; \gamma; q) = \prod_{p|q'} K_{\chi_{p^a}}(\alpha, \beta_{1p}; \gamma; p^{a_p}) \prod_{p|q''} K_{\chi_{p^b}}(\alpha, \beta_{2p}; \gamma; p^{b_p}), \tag{4.2}$$

where

$$\beta_1 \equiv \beta(q'')^{-1} \pmod{q'}, \qquad \beta_2 \equiv \beta(q')^{-1} \pmod{q''},$$

$$\beta_{1p} \equiv \beta \left(\frac{q}{p^{a_p}}\right)^{-1} \pmod{p^{a_p}}, \qquad \beta_{2p} \equiv \beta \left(\frac{q_1}{p^b}\right)^{-1} \pmod{p^b},$$

moreover in the second product all functions χ_{p^b} be the constant function 1.

First we consider the multiples of second product.

Let $\chi_{p^b} = 1$. Write $\gamma = \gamma_1 \gamma_p$, where $(\gamma_1, p) = 1$, $\gamma_p = \gamma_1^{d_1}\gamma_2^{d_2}$ if $p \equiv 1 \pmod 4$, $p = \mathfrak{p}_1\mathfrak{p}_2$; or $\gamma_p = p^d$ if $p \equiv 3 \pmod 4$.

Let $p \equiv 3 \pmod 4$. Using the substitution $y = \gamma_1 y_1$ we obtain for $d < m$

$$K_{\chi_{p^b}}(\alpha, \beta; \gamma; p^m) = K_{\chi_{p^b}}(\alpha, \beta\gamma_1; p^d; p^m).$$

The congruence $xy \equiv p^d \pmod{p^m}$ has the solutions of type $x = p^i x_1$, $y = p^j y_1$, $i + j = d$, $x_1 y_1 \equiv 1 \pmod{p^{m-d}}$. Thus, grouping the summands of sum in $K_{\chi_{p^b}}(\alpha, \beta\gamma_1; p^d; p^m)$, according to $i = \nu_p(x) \le d$, we infer (for χ_{p^d} being constant function 1)

$$K_{\chi_{p^b}}(\alpha, \beta\gamma_1; p^d; p^m) = \sum_{i=0}^{d} \sum_{x \in G^*_{p^{m-d}}} \sum_{x_1 \in G^*_{p^{d-i}}} \sum_{y_1 \in G^*_{p^i}} e_{p^m}(\alpha p^i(x + p^{m-k}x_1) + \beta p^{d-i}(x^{-1} + p^{m-k}y_1)).$$

Now, the summations over x_1, y_1 give nonezero only for $\alpha \equiv 0 \pmod{p^{d-i}}$ and $\beta \equiv 0 \pmod{p^i}$.

We have therefore obtained the following expression (if $\chi_{p^d} = 1$):

$$K_{\chi_{p^b}}(\alpha, \beta\gamma_1; \gamma_p; p^m) = \begin{cases} N(p)^d \sum_{i=I_1}^{I_2} K\left(\dfrac{\alpha}{p^{d-i}}, \dfrac{\beta\gamma_1}{p^i}; p^{m-d}\right) & \text{if} \quad d \le \nu_p(\alpha) + \nu_p(\beta), \\ 0 & \text{otherwise}, \end{cases} \tag{4.3}$$

where $I_1 = k - \nu_p(\alpha)$, $I_2 = \nu_p(\beta)$.

For $p \equiv 1 \pmod 4$ we take into account that χ_{p^d} is the constant function 1 and then

$$K_{\chi_{p^b}}(\alpha, \beta\gamma_1; \gamma_p; p^m) = K\left(\alpha\overline{\mathfrak{p}}_2^m, \beta\gamma_1\overline{\mathfrak{p}}_2^m; \mathfrak{p}_1^{d_1}; \mathfrak{p}_2^m\right) \cdot K\left(\alpha\overline{\mathfrak{p}}_1^m, \beta\gamma_1\overline{\mathfrak{p}}_1^m; \mathfrak{p}_2^{d_2}; \mathfrak{p}_2^m\right),$$

where $\mathfrak{p}_1\overline{\mathfrak{p}}_1 \equiv 1 \pmod{\mathfrak{p}_2^m}$, $\mathfrak{p}_2\overline{\mathfrak{p}}_2 \equiv 1 \pmod{\mathfrak{p}_1^m}$.

Thus we have, by the same arguments as for $p \equiv 3 \pmod 4$

$$K_{\chi_{p^b}}(\alpha, \beta\gamma_1; \mathfrak{p}_1^{d_1}\mathfrak{p}_2^{d_2}; p^m) =$$

$$= \begin{cases} p^{d_1+d_2} \prod_{j=1}^{2} \sum_{i_1=I_{11}}^{I_{12}} \sum_{i_2=I_{21}}^{I_{22}} \mathfrak{M}_{i_1,i_2} & \text{if} \quad d_j \le \nu_{\mathfrak{p}_j}(\alpha) + \nu_{\mathfrak{p}_j}(\beta), \quad j = 1, 2, \\ 0 & \text{otherwise}, \end{cases} \tag{4.4}$$

where

$$\mathfrak{M}_{k_1,k_2} = K\left(\frac{\alpha}{\mathfrak{p}_1^{d_1-k_1}}, \frac{\beta\gamma_1}{\mathfrak{p}_1^{k_1}}; \mathfrak{p}_1^{m-d_1}\right) \cdot K\left(\frac{\alpha}{\mathfrak{p}_2^{d_2-k_2}}, \frac{\beta\gamma_1}{\mathfrak{p}_2^{k_2}}; \mathfrak{p}_2^{m-d_2}\right),$$

$$I_{j1} = d_1 - \nu_{\mathfrak{p}_j}(\alpha), \qquad I_{j2} = \nu_{\mathfrak{p}_j}(\beta), \quad j = 1, 2.$$

From (4.3) and Lemma 2.2 we obtain

$$\left|K_{\chi_{p^b}}(\alpha, \beta; \gamma; p^m)\right| \le (d+1)(m+1)\sqrt{N(\alpha\gamma, \beta\gamma, p^m)} \cdot N(p^m). \tag{4.5}$$

For $d \ge m$, $p \equiv 3 \pmod 4$ we set

$$x = p^i t_1, \qquad y = p^{m-i} t_2, \qquad t_1 \in G^*_{p^{m-i}}, \quad t_2 \in G_{p^i},$$

and get

$$K_\chi(\alpha, \beta\gamma_1; p^d; p^m) = \sum_{i=0}^{m} \sum_{t_1 \in G^*_{p^{m-i}}} \sum_{t_2 \in G_{p^i}} e_{p^m}\left(\Re\big(\alpha p^i t_1 + \beta\gamma_1 p^{m-i} t_2\big)\right). \tag{4.6}$$

The sum over t_2 is $N(p^i)$ or 0 according to whether $i \le \nu_p(\beta)$ or not. The sum over t_1 is the Ramanujan sum, so that

$$\sum_{t_1 \in G^*_{p^{m-i}}} e_{p^m}(\alpha p^i t_1) = \begin{cases} N(p^{m-i}) - N(p^{m-i-1}) & \text{if} \quad 0 < m-i \le \nu_p(\alpha), \\ -N(p^{m-i-1}) & \text{if} \quad m-i = \nu_p(\alpha)+1, \\ 0 & \text{if} \quad m-i > \nu_p(\alpha)+1. \end{cases}$$

In particular, we see that ith term in (4.6) vanishes unless $m - i \le \nu_p(\alpha) + 1$ and $i \le \nu_p(\beta)$, i.e., $m - \nu_p(\alpha) - 1 \le i \le \nu_p(\beta)$. Thus the whole expression vanishes unless $m \le \nu_p(\alpha) + \nu_p(\beta) + 1$.

Thereby in the case $d \ge m$

$$K_\chi(\alpha, \beta\gamma_1; p^d; p^m) \le (d+1)\left(N\left(\alpha\gamma, \beta\gamma; p^m\right)\right)^{1/2} N\left(p^m\right)^{1/2}. \tag{4.7}$$

For $p \equiv 1 \pmod 4$ we obtain an analogous result.

Let $K_{\chi_{p^a}}(\alpha, \beta; \gamma; p^m)$ be a multiple from the first product in (4.2).

Now, we have $\chi_{p^a}(N(x)) = 0$ if $(x,p) \ne 1$, and hence from $xy \equiv \gamma \pmod{p^m}$, $(x,p) = 1$, follows that $y \equiv x^{-1}\gamma \pmod{p^\ell}$, and also

$$K_{\chi_{p^a}}(\alpha, \beta; \gamma; p^m) = K_{\chi_{p^a}}(\alpha, \beta\gamma; 1; p^m) = K_{\chi_{p^a}}(\alpha, \beta\gamma; p^m) \tag{4.8}$$

holds.

The application of Corollary 3.1 gives

$$|K_\chi(\alpha, \beta; \gamma; p^m)| \le \overline{\tau}(p^m)(N(\alpha, \beta\gamma; p^m))^{1/2}(N(p^m))^{1/2} \tag{4.9}$$

if χ is a Dirichlet character $\bmod\, p^{m_0}$, $0 < m_0 \le m$.

Multiplying the local estimates (4.7) and (4.8) together, by (4.2) we have

$$|K_\chi(\alpha, \beta; \gamma; q)| \le \overline{\tau}(\gamma)\overline{\tau}(q)\sqrt{N(\alpha\gamma, \beta\gamma, q)} \cdot N(p)^{m/2}.$$

So, we proved the following main theorem.

Theorem 4.1. *Let α, β, γ be the Gaussian integers, $q > 1$ be a positive integer, χ be a Dirichlet character modulo q_1, $q_1|q$. Then the estimate*

$$|K_\chi(\alpha, \beta; \gamma; q)| \leq \overline{\tau}(\gamma)\overline{\tau}(q)\sqrt{N(\alpha\gamma, \beta\gamma, q)}q$$

holds.

***Remark* 4.1.** The method we used to prove the theorem may be applied in the case of q is even. It is enough to apply the analogues of Lemmas 2.1 and 2.3.

***Remark* 4.2.** In [9] Knightly and Li have shown that the generalized twisted Kloosterman sum over $\mathbb{Z}$ has the estimate

$$|K_\chi(a, b; n; q)| := \left| \sum_{\substack{x,y \in \mathbb{Z}_q \\ xy \equiv n \pmod{q}}} \chi(x) e^{2\pi i \frac{ax+by}{q}} \right| \leq \tau(n)\tau(q)(an, bn, q)^{1/2} q^{1/2} q_\chi^{1/2},$$

where χ is a Dirichlet character modulo q_1 of conductor q_χ, $q_1 \mid q$.

Using the same method of proof as above, we can obtain more precise estimate

$$|K_\chi(a, b; n; q)| \leq \tau(n)\tau(q)(an, bn, q)^{1/2} q^{1/2}.$$

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