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PERIODIC AND BOUNDED SOLUTIONS OF THE COULOMB EQUATION OF MOTION OF TWO AND THREE POINT CHARGES WITH EQUILIBRIUM ON LINE

ПЕРІОДИЧНІ ТА ОБМЕЖЕНІ РОЗВ'ЯЗКИ РІВНЯНЬ РУХУ КУЛОНА ДЛЯ ДВОХ ТА ТРЬОХ ТОЧКОВИХ ЗАРЯДІВ З РІВНОВАГОЮ НА ПРЯМІЙ

Periodic and bounded solutions of the Coulomb equation of motion on line for two and three identical negative point charges in fields of two and three fixed point charges located symmetrically are found. The systems possess equilibrium configurations. The Lyapunov, Siegel, Moser, and Weinstein theorems are applied.

Знайдено періодичні та обмежені розв'язки рівнянь руху Кулона на прямій для двох та трьох однакових негативних точкових зарядів у полі двох та трьох точкових фіксованих зарядів, що симетрично розташовані. Ці системи мають рівноважні стани. При цьому використано теореми Ляпунова, Зігеля, Мозера та Вайнстайна.

1. Introduction. Construction of solutions of classical electrodynamics, based on the Maxwell–Lorentz (ML) equations for point charges, is a fundamental task of mathematics. The simplest approximations of these equations are the Coulomb and Darwin equations which do not take into account radiation of point charges. Solutions of the former and latter were proven to exist on a finite time interval on which there are no collisions in [1] and [17]. If one finds Coulomb systems of point charges with equilibrium then it will be possible to apply several basic theorems, including the center Lyapunov theorem, guaranteeing existence of solutions of the equation of motion on all the time interval. It would be important also to establish equilibrium stability or instability.

In the system of three point charges without external fields one equilibrium configuration is well known [2]: the three charges are located on a line, two of them have the same value $-e_0$, the third charge is placed between them at the same distance and has the value $\frac{e_0}{4}$. The rigourously proved Earnshaw theorem [2, 3] establishes instability of equilibrium in the Coulomb point charges systems without external fields. There is no such result for the systems in an external field created by fixed charges (charged centers). For Coulomb systems of point charges on a line in external fields, created by fixed symmetric charges outside of the line, equilibrium may be stable. The simplest of them will be considered by us here.

In this paper we consider Coulomb systems of two and three point charges with all masses equal to m restricted to move along a line where there is an equilibrium. They are the following ones:

A. Two charges with the same value $-e_0 < 0$ in the field of two symmetric point charges with the value $e' > 0$. The latter are placed at the perpendicular, crossing the origin, at the same distance $b > 0$ from the line where the negative point charges move.

B. The system A with the additional negative charge $-e_0$ which is immobile at the origin (an additional repulsive center for two moving charges is introduced).

C. The system of three equal negative charges $-e_0$ in the field of the two attracting positive charges the same as in the system A.

D. The system of three point charges $-e_0$, $-e_0$, $\frac{e_0}{4}$.

The systems A and B are the most simple Coulomb systems with equilibrium. They are also restricted systems meaning that their dynamics describes the dynamics of two identical charges on a plane with the attracting centers there if their initial velocities (momenta) are directed along the line where the two negative charges are placed initially (at the same distance from the origin for the system B). The system B is derived from the system C since the negative charge at the origin of the latter is immobile if its initial velocity (momentum) is zero and the other two charges start to move at the same distance from the origin with same velocities.

We prove the existence of bounded and periodic solutions for the systems A, B, C with the help of the semilinearization Siegel and Lyapunov, Moser, Weinstein theorems, respectively. The first three theorems demand a resonance condition on $\frac{e_0}{e'}$ and the Weinstein theorem does not. The latter can be applied only for a stable equilibrium since it demands a Hamiltonian to have a Hessian with positive eigenvalues. We prove also the existence of bounded solutions for the systems with the help of the analog of the Siegel theorem which does not demand the resonance condition. All these theorems are formulated in the Appendix.

We prove in the fifth section that the matrix of second derivatives of the potential energy (its Hessian) for the system D at its equilibrium has zero eigenvalues that are obstructions for an application of the above theorems for the system D.

We prove the existence of bounded solutions relying upon the following general theorem which is proved in the Appendix. It follows from the semilinearization Siegel theorem and its mentioned analog (Theorems 6.1 and 6.2).

Theorem 1.1. *Let the Hessian of a potential energy U of a mechanical system of n bodies on a line with equal masses have nonzero eigenvalues $m\sigma_j$ at an equilibrium x_j^0, $j = 1, \ldots, n$, U be holomorphic at it and $\sigma_j < 0$, $j = 1, \ldots, p$. Let also either $\lambda_j = -\sqrt{-\sigma_j}$, $j = 1, \ldots, p < n$, be not in resonance or $p = n$. Then the equation of motion*

$$m\frac{d^2 x_j}{dt^2} = -\frac{\partial U(x_{(n)})}{\partial x_j}, \quad j = 1, \ldots, n, \quad x_{(n)} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{1.1}$$

admits bounded solutions which are holomorphic functions at the origin in p real parameters and $\|x - x^0\|_\lambda < \infty$, $\|\dot{x}\|_\lambda < \infty$, where

$$\|x\|_\lambda = \sup_{t \geq 0} \max_{j \in (1, \ldots, n)} e^{\lambda t} \max |x_j(t)|, \quad \lambda < \lambda_0 = \min_{j=1,\ldots,p} |\lambda_j|.$$

The existence of periodic solutions for mechanical systems with equilibria determined by nondegenerate minima is obtained by us with the help of the following theorem which follows from the Weinstein theorem (Theorem 6.3) in a straightforward fashion.

Theorem 1.2. *Let the potential energy U of a mechanical system of n bodies on a line with equal masses have an equilibrium at $x_{(n)}^0$, be a holomorphic function in its neighborhood and its Hessian determine a positive definite quadratic form in a neighborhood of the equilibrium. Then there exist n periodic solutions of* (1.1) *belonging to the energy level $E_h = U(x_{(n)}^0) + h$ with a sufficiently small positive h whose periods are close to those of the linearized equation of motion.*

These theorem can be generalized to n-body mechanical systems with different masses. The next four sections of our paper are devoted to the above four systems A – D. In the end of each of them, except the fifth section devoted to the D system, we formulate our results in theorems concerning stability of their equilibria and existence of periodic an bounded solutions of the Coulomb equation of motion.

Note that since in the systems A – C the order of charges is preserved due to the infinite repulsion we can substitute the holomorphic functions $(x_j - x_k)^{-1}$ instead of $|x_j - x_k|^{-1}$ and for the system B x_j^{-1} instead of $|x_j|^{-1}$ in the expression for their potential energies.

2. Two charges and two attracting centers. The potential energy for the system A is given by

$$U(x_{(2)}) = e_0^2|x_1 - x_2|^{-1} - 2e_0e'\left(\sqrt{x_1^2 + b^2}\right)^{-1} - 2e_0e'\left(\sqrt{x_2^2 + b^2}\right)^{-1}, \quad x_j \in \mathbb{R}. \tag{2.1}$$

The equilibrium equations are given by $\dfrac{\partial}{\partial x_j}U(x_{(2)}) = 0$, $j = 1, 2$. Let's insert the equalities into it for $k = 1$

$$\frac{\partial}{\partial x_1}|x_1 - x_2|^{-k} = -k\frac{x_1 - x_2}{|x_1 - x_2|^{k+2}}, \quad \frac{\partial}{\partial x_1}\left(\sqrt{x_1^2 + b^2}\right)^{-k} = -k\frac{x_1}{(\sqrt{x_1^2 + b^2})^{k+2}}.$$

That is

$$\frac{\partial}{\partial x_j}U(x_{(2)}) = -e_0^2\frac{x_j - x_k}{|x_1 - x_2|^3} + 2e_0e'\frac{x_j}{\left(\sqrt{x_1^2 + b^2}\right)^3}, \quad j \neq k = 1, 2.$$

As a result we obtain the equilibrium relation putting $x_1 = x_1^0 = a$, $x_2 = x_2^0 = -a$

$$\frac{e_0}{(2a)^3} = \frac{e'}{(\sqrt{a^2 + b^2})^3}.$$

The most important information is the property of the matrix of second derivatives of U. Its two nondiagonal elements are easily calculated as

$$\frac{\partial U(x_{(2)})}{\partial x_1 \partial x_2} = \frac{\partial U(x_{(2)})}{\partial x_2 \partial x_1} = -2e_0^2|x_1 - x_2|^{-3}.$$

Let $U_{1,2}^0$ be this function at the equilibrium. Then $U_{1,2}^0 = -\dfrac{e_0^2}{4a^3} = -u'$. Further

$$\frac{\partial^2}{\partial x_j^2}U(x_{(2)}) = \frac{2e_0^2}{|x_1 - x_2|^3} + \frac{2e_0e'}{\left(\sqrt{x_j^2 + b^2}\right)^3} - \frac{6e_0e'x_j^2}{\left(\sqrt{x_j^2 + b^2}\right)^5}.$$

Let $U_{j,j}^0$ be this function at the equilibrium. Then

$$U_{1,1}^0 = U_{2,2}^0 = \frac{e_0^2}{4a^3} + \frac{2e_0e'}{(\sqrt{a^2 + b^2})^3} - \frac{6e_0e'a^2}{(\sqrt{a^2 + b^2})^5}.$$

From the equilibrium relation it follows that

$$\left(\frac{e_0}{e'}\right)^{1/3}\frac{1}{2a} = \frac{1}{\sqrt{a^2 + b^2}}, \quad a = (4 - \eta)^{-1/2}\sqrt{\eta}b, \quad \eta = \left(\frac{e_0}{e'}\right)^{2/3} < 4. \tag{2.2}$$

As a result

$$U(x^0_{(2)}) = U(-a,a) = \frac{e_0^2}{2a} - \frac{2e_0e'}{a}\left(\frac{e_0}{e'}\right)^{1/3} = \frac{e_0^2}{2a}\left(1 - 4\left(\frac{e'}{e_0}\right)^{2/3}\right),$$

$$U^0_{1,1} = U^0_{2,2} = 2u' - u_*, \quad u_* = \frac{3e_0^2}{4^2a^3}\left(\frac{e_0}{e'}\right)^{2/3} = \frac{3u'}{4}\left(\frac{e_0}{e'}\right)^{2/3}.$$

Let U^0 be the matrix with the elements $U^0_{1,1}$, $U^0_{2,2}$, $U^0_{1,2} = U^0_{2,1}$. Its eigenvalues ζ_1, ζ_2 are found easily as the roots $\lambda = \zeta_1, \zeta_2$ of the equation

$$\begin{gathered}(2u' - u_* - \lambda)^2 - u'^2 = 0,\\ \zeta_1 = u' - u_*, \quad \zeta_2 = 3u' - u_*.\end{gathered} \tag{2.3}$$

Two eigenvalues are negative or positive if $3u' < u_*$, $u' > u_*$, respectively. The following result follows from the stability Lagrange – Dirichlet theorem [3] since the latter case means that the potential energy attains a minimum at the equilibrium.

Theorem 2.1. *The line system of two identical charges with the the potential energy* (2.1) *possesses the equilibrium* $x^0 = (-a, a)$ *with* a *given by* (2.2) *if* $\frac{e_0}{e'} < 8$ *and it is stable if*

$$\frac{e_0}{e'} < \frac{8}{\sqrt{27}} = \left(\frac{4}{3}\right)^{3/2}.$$

If the neutrality condition $e' = e_0$ *holds then the equilibrium is stable.*

If $3u' > u_*$, $u' < u_*$ then one root is positive and the other is negative. If $u' = u_*$ or $3u' = u_*$ then one root is positive or negative and the other is zero.

The linear part of the vector field at the equilibrium of the Coulomb equation of motion has the eigenvalues $\pm i\sqrt{m^{-1}(3u' - u_*)}$, $\pm i\sqrt{m^{-1}(u' - u_*)}$ if $u' > u_*$. In order to apply the Lyapunov theorem in this case one has to exclude the resonance between ζ_1, ζ_2, i.e., $\frac{\zeta_2}{\zeta_1} = k^2$, $1 < k \in \mathbb{Z}$.

Theorem 2.2. *Let* $\frac{e_0}{e'} < \frac{8}{\sqrt{27}}$ *and* $\frac{e_0}{e'} \neq \frac{8}{\sqrt{27}}\left(\frac{k^2-3}{k^2-1}\right)^{3/2}$, $1 < k \in \mathbb{Z}^+$ *or* $\frac{8}{\sqrt{27}} < \frac{e_0}{e'} < 8$. *Then the Coulomb equation of motion* (1.1) *for* $n = 2$ *and potential energy* (2.1) *possesses two or one periodic solutions such that each of them depends on a real parameter* c_j *for some* j, $j = 1, 2$, *or one real parameter* c. *These solutions and their periods* $\tau_1(c_1), \tau_2(c_2)$ *or* $\tau(c)$ *are holomorphic functions in the parameters at the origin and* $\tau_1(0) = 2\pi\sqrt{m}\left(u'\left(1 - \frac{3}{4}\left(\frac{e_0}{e'}\right)^{2/3}\right)\right)^{-1/2}$, $\tau_2(0) = 2\pi\sqrt{m}\left(u'\left(3 - \frac{3}{4}\left(\frac{e_0}{e'}\right)^{2/3}\right)\right)^{-1/2}$ *or* $\tau(0) = 2\pi\sqrt{m}\left(u'\left(3 - \frac{3}{4}\left(\frac{e_0}{e'}\right)^{2/3}\right)\right)^{-1/2}$.

The next theorem follows from the Theorem 1.2.

Theorem 2.3. *Let* $\frac{e_0}{e'} < \frac{8}{\sqrt{27}}$. *Then the Coulomb equation of motion* (1.1) *for* $n = 2$ *and potential energy* (2.1) *on the energy level* $E_h = \frac{e_0^2}{2a}\left(1 - 4\left(\frac{e'}{e_0}\right)^{2/3}\right) + h$ *with a sufficiently small positive*

h has two periodic solutions with the frequencies close to the square roots of $m^{-1}\left(3-\frac{3}{4}\left(\frac{e_0}{e'}\right)^{2/3}\right)u'$, $m^{-1}\left(1-\frac{3}{4}\left(\frac{e_0}{e'}\right)^{2/3}\right)u'$.

From the Theorem 1.1 one deduces the following theorem which corresponds to the case of nonnegative roots in (2.3).

Theorem 2.4. *Let* $\frac{8}{\sqrt{27}}<\frac{e_0}{e'}<8$. *Then the Coulomb equation of motion* (1.1) *for* $n=2$ *and the potential energy* (2.1) *admits bounded solutions which are holomorphic functions at the origin in one real parameter such that* $\|x-x^0\|_\lambda<\infty$, $\|\dot{x}\|_\lambda<\infty$, *where* $\lambda<\sqrt{m^{-1}(u_*-u')}$ *and* x^0 *is the equilibrium.*

The next theorem is derived from the Moser theorem (Theorem 6.5).

Theorem 2.5. *Let* $\frac{8}{\sqrt{27}}<\frac{e_0}{e'}<8$. *Then the Coulomb equation of motion* (1.1) *for* $n=2$ *and the potential energy* (2.1) *admits bounded solutions which are holomorphic functions at the origin in three real parameters* c_j, $j=1,2,3$. *If* $c_3=0$ *these solutions are periodic functions with the period* $\tau(c_1c_2)$ *which coincides at the zero with the period from the Theorem* 2.2.

The neutrality condition implies the resonance condition with $k=3$ in the Theorem 2.2 and this means that if the neutrality condition $e_0=e'$ holds then the conclusion of the Theorem 2.3 is true.

3. Two negative charges and three centers. The system B is characterized by the potential energy

$$U(x_{(2)})=\frac{e_1e_2}{|x_1-x_2|}-2e'e_0\sum_{k=1}^{2}\left(\sqrt{x_k^2+b^2}\right)^{-1}+e_0^2\sum_{k=1}^{2}|x_k|^{-1},\quad -e_1=-e_2=e_0>0,\quad e'>0. \tag{3.1}$$

As a result

$$\frac{\partial}{\partial x_j}U(x_{(2)})=-e_1e_2\frac{x_j-x_k}{|x_1-x_2|^3}+\frac{2e'e_0x_j}{\left(\sqrt{x_j^2+b^2}\right)^3}-\frac{e_0^2x_j}{|x|^3},\quad j\neq k=1,2.$$

The equilibrium is given by $x_1^0=a$, $x_2^0=-a$. The equilibrium relation is determined by

$$\frac{5e_0^2}{4a^2}-\frac{2e'e_0a}{(\sqrt{a^2+b^2})^3}=0,$$

which follows from $\frac{\partial}{\partial x_j}U(x_{(2)})=0$, $j=1,2$. The matrix of second derivatives of U is easily calculated as

$$\frac{\partial^2}{\partial x_j^2}U(x_{(2)})=2e_1e_2|x_1-x_2|^{-3}+\frac{2e_0e'}{\left(\sqrt{x_j^2+b^2}\right)^3}-\frac{6e_0e'x_j^2}{\left(\sqrt{x_j^2+b^2}\right)^5}+\frac{2e_0^2}{|x_j|^3},$$

$$\frac{\partial^2U(x_{(2)})}{\partial x_1\partial x_2}=\frac{\partial^2U(x_{(2)})}{\partial x_2\partial x_1}=-2e_1e_2|x_1-x_2|^{-3},$$

$$U_{2,1}^0=U_{1,2}^0=-u'=-\frac{e_0^2}{4a^3},$$

$$U^0_{1,1} = U^0_{2,2} = 9u' + \frac{2e_0e'}{(\sqrt{a^2+b^2})^3} - \frac{6e_0e'a^2}{(\sqrt{a^2+b^2})^5}.$$

From the equilibrium relation one derives

$$\frac{2e_0e'}{(\sqrt{a^2+b^2})^3} = 5u', \quad (\sqrt{a^2+b^2})^{-1} = \frac{1}{2a}\left(\frac{5e_0}{e'}\right)^{1/3}, \tag{3.2}$$

$$a = (4-\eta')^{-1/2}\sqrt{\eta'}b, \quad \eta' = \left(\frac{5e_0}{e'}\right)^{2/3} < 4, \quad \frac{e_0}{e'} < \frac{8}{5}. \tag{3.3}$$

That is

$$U(x^0_{(2)}) = U(-a,a) = \frac{5e_0^2}{2a} - \frac{2e_0e'}{a}\sqrt{\eta'},$$
$$U^0_{1,1} = U^0_{2,2} = 14u' - u_*, \quad u_* = \frac{3e_0^2}{4^2a^3}\eta' = \frac{3u'}{4}\eta' < 3u', \tag{3.4}$$

where $U^0_{j,k}$ are the values of partial derivatives of the potential energy at the equilibrium. The neutrality condition is $2e' = 3e_0$ which implies $u_* = 3\left(\frac{10}{3}\right)^{2/3} u'$. Let U^0 be the matrix with the elements $U^0_{1,1}$, $U^0_{2,2}$, $U^0_{1,2} = U^0_{2,1}$. Its eigenvalues ζ_1, ζ_2 are found easily as the roots $\lambda = \zeta_1, \zeta_2$ of the equation

$$(14u' - u_* - \lambda)^2 - u'^2 = 0,$$

$$\zeta_1 = 13u' - u_*, \quad \zeta_2 = 15u' - u_*.$$

We have $\zeta_j > 0$ since $u_* < 3$ and the case of two negative roots is excluded. As a result we can formulate only the analogs of the Theorems 2.1 and 2.2.

Theorem 3.1. *The line system of two identical charges with the the potential energy* (3.1) *possesses the equilibrium $x^0 = (-a,a)$ with a given by* (3.2) *which is stable if*

$$\frac{e_0}{e'} < \frac{8}{5}.$$

If the neutrality condition $2e' = 3e_0$ holds then the equilibrium is stable.

Since $k^2 \neq \left(15 - \frac{3}{4}\left(\frac{5e_0}{e'}\right)^{2/3}\right)\left(13 - \frac{3}{4}\left(\frac{5e_0}{e'}\right)^{2/3}\right)^{-1} < \frac{3}{2}$, $k \in \mathbb{Z}^+$ for $\frac{e_0}{e'} < \frac{8}{5}$ the following theorem follows from the Lyapunov theorem (Theorem 6.4).

Theorem 3.2. *Let $\frac{e_0}{e'} < \frac{8}{5}$. Then the Coulomb equation of motion* (1.1) *for $n = 2$ with the potential energy* (3.1) *possesses two periodic solutions such that each of them depends on a real parameter c_j for some j, $j = 1, 2$. These solutions and their periods $\tau_1(c_1)$, $\tau_2(c_2)$ are holomorphic functions in the parameters at the origin and* $\tau_1(0) = 2\pi\sqrt{m}\left(u'\left(13 - \frac{3}{4}\left(\frac{5e_0}{e'}\right)^{2/3}\right)\right)^{-1/2}$, $\tau_2(0) = 2\pi\sqrt{m}\left(u'\left(15 - \frac{3}{4}\left(\frac{5e_0}{e'}\right)^{2/3}\right)\right)^{-1/2}$.

Note that if the neutrality condition $2e' = 3e_0$ holds then the conclusion of the Theorem 3.2 is true.

4. Three charges and two attracting centers. The potential energy of the system C is written as follows:

$$U(x_{(3)}) = \frac{1}{2}\sum_{j\neq k=1}^{3}\frac{e_j e_k}{|x_j - x_k|} - 2e'e_0\sum_{k=1}^{3}\left(\sqrt{x_k^2+b^2}\right)^{-1}, \quad -e_j = e_0 > 0, \quad e' > 0. \qquad (4.1)$$

The matrix of second derivatives of the potential energy is obtained from the equalities

$$\frac{\partial}{\partial x_j}U(x_{(3)}) = -e_j\sum_{k=1,k\neq j}^{3}e_k\frac{x_j - x_k}{|x_j - x_k|^3} + \frac{2e'e_0x_j}{\left(\sqrt{x_j^2+b^2}\right)^3},$$

$$\frac{\partial^2}{\partial x_j^2}U(x_{(3)}) = 2e_j\sum_{k=1,k\neq j}^{3}e_k|x_j - x_k|^{-3} + \frac{2e_0e'}{\left(\sqrt{x_j^2+b^2}\right)^3} - \frac{6e_0e'x_j^2}{\left(\sqrt{x_j^2+b^2}\right)^5},$$

$$\frac{\partial^2 U(x_{(3)})}{\partial x_j\partial x_k} = \frac{\partial^2 U(x_{(3)})}{\partial x_k\partial x_j} = -2e_je_k|x_j - x_k|^{-3}.$$

The equilibrium is given by $x_1 = x_1^0 = -a$, $x_2 = x_2^0 = 0$, $x_3 = x_3^0 = a$. The first derivative of the potential energy for $j = 2$ is zero. The equilibrium relation

$$\frac{5e_0^2}{4a^2} - \frac{2e'e_0a}{(\sqrt{a^2+b^2})^3} = 0$$

follows from $\dfrac{\partial}{\partial x_j}U(x_{(3)}) = 0$, $j = 1, 3$, and coincides with the equilibrium relation from the previous section. At the equilibrium the matrix of second derivatives of the potential energy is deduced from the equalities

$$U_{2,1}^0 = U_{1,2}^0 = U_{2,3}^0 = U_{3,2}^0 = -\frac{2e_0^2}{a^3} = -8u', \quad U_{3,1}^0 = U_{1,3}^0 = -u',$$

$$U_{1,1}^0 = U_{3,3}^0 = 9u' + \frac{2e_0e'}{(\sqrt{a^2+b^2})^3} - \frac{6e_0e'a^2}{(\sqrt{a^2+b^2})^5}, \quad U_{2,2}^0 = 16u' + \frac{2e_0e'}{(\sqrt{a^2+b^2})^3}.$$

From (3.2)–(3.4) it follows that

$$U_{2,2}^0 = 21u', \quad U_{1,1}^0 = U_{3,3}^0 = 14u' - u_*.$$

Neutrality condition is $2e' = 3e_0$ and it implies $u_* = \dfrac{3}{4}\left(\dfrac{10}{3}\right)^{2/3}u'$.

Let's calculate the diagonal k-dimensional determinants $\mathrm{Det}_kU' = |U'|_k$, $k = 1, 2$, of the matrix $u'^{-1}U^0$ and apply the Sylvester condition for the matrix to be positive definite putting $u = u_*u'^{-1} <$
< 3

$$|U'|_1 = 14 - u > 0, \quad |U'|_2 = U_{1,1}'U_{2,2}' - U_{1,2}'U_{2,1}' = 21(14 - u) - 64 = 230 - 21u > 0,$$

$$\mathrm{Det}\, U' = |U'| = U'_{1,1}(U'_{2,2}U'_{3,3} - U'_{3,2}U'_{2,3}) - U'_{1,2}(U'_{2,1}U'_{3,3} - U'_{2,3}U'_{3,1}) +$$

$$+ U'_{1,3}(U'_{2,1}U'_{3,2} - U'_{2,2}U'_{3,1}) = (14-u)[21(14-u) - 64] + 8[-8(14-u) - 8] -$$

$$-(64+21) = 21(14-u)^2 - 128(14-u) - 149.$$

The roots of the last equation are given by the following remarkable equality:

$$14 - u = 64 \pm \sqrt{64^2 + 21 \times 149} = 64 \pm 85.$$

Hence $\mathrm{Det}\, U' > 0$ if $14 - u > 149$ or $14 - u < -21$. But u is positive and $14 - u \leq 14$ which contradicts the first inequality. The second inequality contradicts the inequality $|U'|_1 > 0$. That is the matrix U^0 is not positive definite. If we find positive roots and there are no zero roots then we can apply the center Lyapunov theorem and find periodic solutions.

The characteristic polynomial $p'(\lambda) = -|U' - \lambda I| = \mathrm{Det}\,(\lambda - U')$ is given by

$$\begin{gathered} -p'(\lambda) = (14 - u - \lambda)[(21 - \lambda)(14 - u - \lambda) - 64] + \\ +8[-8(14 - u - \lambda) - 8] - (64 + 21 - \lambda) = \\ = (21 - \lambda)(14 - u - \lambda)^2 - 128(14 - u - \lambda) - 149 + \lambda, \\ p'(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \\ a_1 = -2(14 - u) - 21, \quad a_2 = 42(14 - u) + (14 - u)^2 - 129, \\ a_3 = -21(14 - u)^2 + 128(14 - u) + 149. \end{gathered} \tag{4.2}$$

Let us put

$$p = a_2 - \frac{a_1^2}{3} = -\frac{1}{3}(14 - u)^2 + 14(14 - u) - 276, \quad q = a_3 - \frac{a_1 a_2}{3} + \frac{2a_1^3}{27}.$$

It is easy to see that $p < 0$. The condition

$$\frac{q^2}{4} + \frac{p^3}{27} \leq 0$$

implies that all the roots are real, as it follows from the Cardano formulas [15], and it holds in our case since we deal with the symmetric matrix U^0. To have all the roots different one has to assume

$$\frac{q^2}{4} + \frac{p^3}{27} < 0. \tag{4.3}$$

If it is true then it is possible to define the angle φ as in [15]

$$\cos\varphi = -\frac{q\sqrt{27}}{2\sqrt{|p|^3}}, \quad 0 < \varphi < \pi.$$

Then the roots $\lambda = \zeta'_k, k = 1, 2, 3,$ of $p'$are written due to the Cardano formulas as

$$\zeta'_k = 2\sqrt{3^{-1}|p|}\cos\frac{\varphi + 2\pi(k-1)}{3}, \quad k = 1, 2, 3.$$

If $p(\lambda)$ is the characteristic polynomial of U^0 then $p(\lambda) = u'^3 p'(u'^{-1}\lambda)$ and the eigenvalues of U^0 are given by $\zeta_k = u'\zeta'_k$ that is

$$\zeta_k = 2u'\sqrt{3^{-1}|p|}\cos\frac{\varphi + 2\pi(k-1)}{3}, \quad k = 1, 2, 3. \tag{4.4}$$

To apply the theorems from the introduction we have to single out cases when the different roots have equal signs and guarantee absence of resonances. It is not difficult to check that

1) $\zeta_1 > 0$, $\zeta_3 > 0$, $\zeta_2 < 0$ if $3^{-1}\varphi \in \left(0, \frac{\pi}{2}\right)$, $3^{-1}(\varphi + 4\pi) \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, that is $3^{-1}\varphi \in$ $\in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$,

2) $\zeta_1 > 0$, $\zeta_2 < 0$, $\zeta_3 < 0$ if $3^{-1}\varphi \in \left(0, \frac{\pi}{6}\right)$.

Moreover in the second case $\frac{\zeta_3}{\zeta_2} \neq k^2, k \in \mathbb{Z}^+$, since $\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$, $\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\cos\frac{\varphi + 2\pi}{3} \in \left[-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right]$. Besides $\cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) = \cos\frac{3\pi}{2} = 0$ and $\zeta_3 = 0$ at $\varphi = \frac{\pi}{2}$. The same argument excludes also the resonance $\frac{\zeta_3}{\zeta_1} \neq k^2$, $k \in \mathbb{Z}^+$, in the first case since $\cos\frac{\varphi}{3} \in \left[-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right]$.

The first result of this section is formulated in the following theorem deduced from the Lyapunov theorem (the case 1 demands the nonresonance condition in ζ_1, ζ_3 for the existence of the two-parametric family of periodic solutions). The Theorem 1.2 can not be applied since not all ζ_j and σ_j can be positive.

Theorem 4.1. *If* (4.3) *holds and* $3^{-1}\varphi \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ *and* $\left(\cos\frac{\varphi + 4\pi}{3}\right)^{-1}\cos\frac{\varphi}{3} \neq k^2$, $k \in \mathbb{Z}^+$, *then the Coulomb equation of motion* (1.1) *for* $n = 3$ *with the potential energy* (4.1) *possesses periodic solutions such that each of them depends on a real parameter* c_j *for some* j, $j = 1, 2$. *These solutions and their periods* $\tau_1(c_1), \tau_2(c_2)$ *are holomorphic functions in the parameters at the origin and* $\tau_1(0) = 2\pi\sqrt{\frac{m}{\zeta_1}}$, $\tau_2(0) = 2\pi\sqrt{\frac{m}{\zeta_3}}$, *where* ζ_j *are determined by* (4.4). *If* $3^{-1}\varphi \in \left(0, \frac{\pi}{6}\right)$ *then the same equation possesses a periodic solution depending on a constant* c. *This solution and its period* $\tau(c)$ *are holomorphic functions in the parameter at the origin and* $\tau(0) = \tau_1(0)$.

From the Theorem 1.1 we deduce the following result since (the case 2 demands the non-resonance condition in ζ_2, ζ_3 for the existence of the two-parametric family of bounded solutions).

Theorem 4.2. *Let* (4.3) *hold and either* $3^{-1}\varphi \in \left(0, \frac{\pi}{6}\right)$ *and* $\left(\cos\frac{\varphi + 2\pi}{3}\right)^{-1}\cos\frac{\varphi + 4\pi}{3} \neq$ $\neq k^2$, $k \in \mathbb{Z}^+$, *or* $3^{-1}\varphi \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right)$. *Then the Coulomb equation of motion* (1.1) *for* $n = 3$ *with the potential energy* (4.1) *admits bounded solutions which are holomorphic functions at the origin either in two or one real parameters such that* $\|x - x^0\|_\lambda < \infty$, $\|\dot{x}\|_\lambda < \infty$, *where* $\lambda < \min_{k=2,3}\sqrt{m^{-1}|\zeta_k|}$ *or* $\lambda < \sqrt{m^{-1}|\zeta_2|}$, *the numbers* ζ_k *are given by* (4.4) *and* x^0 *is the equilibrium.*

In order to apply the Moser theorem (Theorem 6.5) we have to guarantee absence of quadratic resonances between ζ_1, ζ_3 and ζ_2, ζ_3 for $3^{-1}\varphi \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ and $3^{-1}\varphi \in \left(0, \frac{\pi}{6}\right)$, respectively. Taking into account that $\frac{\zeta_3}{\zeta_1} \neq k^2, k \in \mathbb{Z}^+$, and $\frac{\zeta_3}{\zeta_2} \neq k^2$, $k \in \mathbb{Z}^+$, for such the cases we derive the following consequence of the Moser theorem.

Theorem 4.3. *If* (4.3) *is true and* $q \neq 0$ *then the Coulomb equation of motion* (1.1) *for* $n = 3$ *with the potential energy* (4.1) *possesses bounded solutions which are holomorphic functions at the origin in three real parameters* c_j, $j = 1, 2, 3$. *If* $c_3 = 0$ *these solutions are periodic functions with the period* $\tau(c_1c_2)$ *which coincides at the zero with the period* $\tau_1(0)$ *from the Theorem* 4.1.

The polynomial of the sixth order in $14 - u$ in the left-hand side of (4.3) can have not more than three real roots of even multiplicities. Lengthy calculations show that it is a polynomial of the fourth order. That is, if the condition (4.3) is violated for real u then it is possible only for one or two values of u.

5. Three charges. The potential energy of the system D is given by

$$U(x_{(3)}) = \frac{1}{2}\sum_{j\neq k=1}^{3} \frac{e_je_k}{|x_j - x_k|}, \quad -e_1 = -e_2 = e_0, \quad e_3 = \frac{e_0}{4} > 0. \tag{5.1}$$

That is

$$U(x_{(3)}) = e_0^2|x_1 - x_2|^{-1} - e_0e_3[|x_1 - x_3|^{-1} + |x_2 - x_3|^{-1}], \quad x_j \in \mathbb{R}.$$

Then

$$\frac{\partial}{\partial x_1}U(x_{(3)}) = -e_0^2\frac{x_1 - x_2}{|x_1 - x_2|^3} + e_0e_3\frac{x_1 - x_3}{|x_1 - x_3|^3},$$

$$\frac{\partial}{\partial x_2}U(x_{(3)}) = -e_0^2\frac{x_2 - x_1}{|x_1 - x_2|^3} + e_0e_3\frac{x_2 - x_3}{|x_2 - x_3|^3},$$

$$\frac{\partial}{\partial x_3}U(x_{(3)}) = -e_0e_3[\frac{x_3 - x_1}{|x_1 - x_3|^3} + \frac{x_3 - x_2}{|x_2 - x_3|^3}].$$

The equality $\frac{\partial}{\partial x_3}U(x_{(3)}) = 0$ holds for $x_1 = x_1^0 = -a$, $x_2 = x_2^0 = a$, $x_3^0 = 0$. This configuration is an equilibrium. This follows also from the equalities $\frac{\partial}{\partial x_j}U(x_{(3)}) = 0$, $j = 1, 2$. It is more convenient to rewrite the derivatives of the potential energy as follows:

$$\frac{\partial}{\partial x_j}U(x_{(3)}) = -e_j\sum_{k=1,k\neq j}^{3} e_k\frac{x_j - x_k}{|x_j - x_k|^3}.$$

Then the second derivatives of the potential energy are calculated as follows:

$$\frac{\partial U(x_{(3)})}{\partial x_j\partial x_k} = \frac{\partial U(x_{(3)})}{\partial x_k\partial x_j} = -2e_je_k|x_j - x_k|^{-3}, \quad k \neq j,$$

$$\frac{\partial^2}{\partial x_j^2}U(x_{(3)}) = 2e_j\sum_{k=1,k\neq j}^{3} e_k|x_j - x_k|^{-3}.$$

Hence the second derivatives of the potential energy at the equilibrium $U^0_{j,k}$ are given by

$$U^0_{1,2} = U^0_{2,1} = -\frac{e_0^2}{4a^3} = -u', \quad U^0_{3,1} = U^0_{1,3} = U^0_{2,3} = U^0_{3,2} = 2u',$$

$$U_{1,1} = U_{2,2} = -u', \quad U_{3,3} = -4u'.$$

The rescaled characteristic polynomial p', i.e., the determinant of the matrix $-U' + \lambda I$, $U' = u'^{-1}U^0$ is given by

$$-p'(\lambda) = (U'_{1,1} - \lambda)[(U'_{2,2} - \lambda)(U'_{3,3} - \lambda) - U'_{3,2}U'_{2,3}] - U'_{1,2}[U'_{2,1}(U'_{3,3} - \lambda) - U'_{2,3}U'_{3,1}] +$$

$$+ U'_{1,3}[U'_{2,1}U'_{3,2} - (U'_{2,2} - \lambda)U'_{3,1}],$$

$$-p'(\lambda) = -(1+\lambda)[(1+\lambda)(4+\lambda) - 4] + (4 + \lambda - 4) + 2[-2 + 2(1+\lambda)] =$$

$$= -\lambda[(1+\lambda)(5+\lambda) - 5] = -\lambda^2(\lambda + 6).$$

Its roots are $\lambda = 0, \lambda = -6$ and the equilibrium is a degenerate maximum of the potential energy. If $p(\lambda)$ is the characteristic polynomial of U^0 then $p(\lambda) = u'^3 p'(u'^{-1}\lambda)$.

Proposition 5.1. *The* (3×3)*-matrix of second derivatives of the potential energy* (5.1) *at its equilibrium has the zero eigenvalue which is doubly degenerate and its third eigenvalue is equal to* $-6u'$.

6. Appdendix. Here we formulate the basic theorems that are applied by us in the previous sections. We begin from the semilinearization Siegel theorem (see Section 28 in [4]).

Theorem 6.1. *Let* $\lambda_j \in \mathbb{C}$,

$$\dot{x}_j(t) = \lambda_j x_j + X_j(x_{(l)}(t)), \quad j = 1, \ldots, l, \quad t \geq 0, \tag{6.1}$$

the function $X_j(x_{(l)})$, $x_{(l)} = (x_1, \ldots, x_l)$, *be holomorphic at the origin such that in their power expansions the sum of powers of* x_j *is not less than two, real parts of* $\lambda_j \in \mathbb{C}$, $j = 1, \ldots, p$, *be negative and the nonresonance condition hold: neither of these p numbers be a linear combination of others with nonnegative integer coefficients whose sum exceeds unity. Then there exist functions* $\varphi'_j(x_{(p)})$, $j = 1, \ldots, p$, $\varphi_j(x_{(p)})$, $j = p+1, \ldots, l$, *which are holomorphic at the origin and zero at it, such that a partial solution of* (1.1) *is given by*

$$x_j(t) = \varphi'_j(e^{\lambda_1 t}c_1, \ldots, e^{\lambda_p t}c_p), \quad j = 1, \ldots, p,$$

$$x_j(t) = \varphi_j(x_{(p)}(t)), \quad j = p+1, \ldots, l,$$

where c_j *are arbitrary constants.*

This theorem for $p = l$ is a version of the linearization Poincare theorem [5, 6] which was generalized by Siegel in his linearization theorem [6, 7]. The statement of the Theorem 6.1 is not formulated as a theorem in [4] and proved in Section 28. The existence of formal series φ_j, φ'_j demands the nonresonance condition which eliminates small denominators and allows one to solve the equation (7) in Section 28 in [4]. The Siegel's proof of the Theorem 6.1 is simpler than the proof of the Poincare linearization theorem given in [5]. The proof of the holomorphic character of φ_j, φ'_j

is based on an application of the standard inequality

$$|\lambda_k - \sum_{j=1}^{p} n_j\lambda_j| > c\sum_{j=1}^{p} n_j, \quad \sum_{j=1}^{p} n_j \geq 2,$$

where n_j is a nonnegative integer, c is a positive constant and a majoration technique.

A resonance version of the Theorem 6.1, which demands that real parts of all λ_j are nonzero, is formulated in the following theorem.

Theorem 6.2. *Let in* (6.1) $\operatorname{Re}\lambda_j < 0,\ j = 1,\ldots,p,\ \operatorname{Re}\lambda_j > 0,\ j = p+1,\ldots,l,\ X_j$ *be a holomorphic function in the hyperdisc*

$$D_l(A) = \{x_j\colon |x_j| \leq A, j = 1,\ldots,l\}, \quad A > 0.$$

$|X_j| \leq M$ *and have the power expansion the same as in the Theorem* 6.1. *Then there exist* $\tilde{x}_0 \in \mathbb{R}^{l-p}$ *and the solution of* (6.1) *for the initial data* $(x_{0(p)}, \tilde{x}_0)$ *with* $x_{0(p)} = x_{(p)}(0) \in D_p(A'),\ A' < A$. *This solution is a holomorphic function in* $x_{0(p)} \in D_p(A')$, *belongs to* $D_l(A)$ *and* $\|x\|_\lambda \leq A,\ \lambda < \lambda_0$ *if* $M(\lambda_0 - \lambda)^{-1}$ *is sufficiently small, where*

$$\|x\|_\lambda = \sup_{t\geq 0}\max_{s\in(1,\ldots,l)} e^{\lambda t}|x_s(t)|, \quad \lambda_0 = \min_{j=1,\ldots,l}|\operatorname{Re}\lambda_j| \quad .$$

Note that $\tilde{x}_0 = (x_{p+1}(0),\ldots,x_l(0))$ is a holomorphic vector function in $x_{0(p)} \in D_p(A')$ and the solutions of (1.1) found in this theorem tend exponentially fast to the equilibrium in the infinite time limit. If F_j is a sufficiently smooth function then the existence of a bounded solution for (1.1) on the positive time interval is proven in [8]. Lyapunov proved this under the assumption $p = l$. This theorem can be proved without difficulty if $\operatorname{Re}\lambda_j \geq 0,\ j = p+1,\ldots,l$, and $X_j(0_{(p)}, x_{p+1},\ldots,x_l) = 0$. The existence of solutions converging to an equilibrium if there are $\lambda_j = 0$ and $X_j \in C^1(\mathbb{R}^d)$ is proved in [18].

Proof of Theorem **1.1.** To prove the theorem one has to transform (1.1) into the simple standard form (6.1) and apply the Theorems 6.1 and 6.2. We assume that the potential energy U has the equilibrium at the points $x_j^0,\ j = 1,\ldots,n$, at which it is holomorphic, that is

$$\left(\frac{\partial U}{\partial x_j}\right)(x_{(n)}^0) = 0.$$

Then in the new variables $x_j' = x_j - x_j^0$ the dynamic equation is rewritten as

$$m\frac{d^2x_j}{dt^2} = -\frac{\partial U'(x_{(n)})}{\partial x_j}, \tag{6.2}$$

where

$$U'(x_{(n)}) = U(x_1 + x_1^0,\ldots,x_n + x_n^0), \quad \left(\frac{\partial U'}{\partial x_j}\right)(0) = 0.$$

Let U^0 be the symmetric matrix of the second derivatives of U calculated at the equilibrium. Then by a nonsingular linear transformation $x_j' = \sum_{k=1}^{n} S'_{j,k}x_k$ one diagonalizes U^0, which has real-valued eigenvalues $m\sigma_j$, that is $\delta_{j,k}\sigma_j = (S'm^{-1}U^0S'^{-1})_{j,k}$ and transforms (6.2) into

$$\frac{d^2x_j}{dt^2} = -\sigma_j x_j(t) + F_j(x_{(n)}), \tag{6.3}$$

where

$$F_j(x_{(n)}) = -m^{-1}\sum_{k=1}^{n} S'_{j,k}\left(\frac{\partial U''}{\partial x_k}\right)((S'^{-1}x)_{(n)}), \quad U''(x_{(n)}) = U'(x_{(n)}) - \frac{1}{2}\sum_{j,k=1}^{n} U^0_{j,k}x_jx_k.$$

That is

$$\frac{dx_j}{dt} = v_j, \quad \frac{dv_j}{dt} = -\sigma_j x_j(t) + F_j(x_{(n)}), \tag{6.4}$$

where F_j has the properties of X_j. Then by the linear two dimensional transformation produced by the matrix S^0_j the last equation is mapped into (6.1) with $l = 2n$ and $-\lambda_{2j-1} = \lambda_{2j} = \sqrt{-\sigma_j}$, $j = 1, \ldots, n$. The matrix S^0_j diagonalizes the two dimensional matrix A_j, which determines the linear part of (6.4), with the zero diagonal elements and nondiagonal elements $A_{j;1,2} = 1$, $A_{j;2,1} = -\sigma_j$. That is $S^0_jA_j = \hat{\sigma_j}S^0_j$, where $\hat{\sigma_j}$ is a diagonal matrix with the eigenvalues $-\lambda_{2j-1} = \lambda_{2j} = \sqrt{-\sigma_j}$. It is not difficult to check that

$$S^0_{j;1,1} = S^0_{j;2,1} = \frac{1}{2}, \quad -S^0_{j;1,2} = S^0_{j;2,2} = \frac{1}{2\kappa_j}, \quad \kappa_j = \sqrt{-\sigma_j}.$$

The new variables look like

$$x'_{2j-1} = \frac{1}{2}\left(x_j - \frac{1}{\kappa_j}v_j\right), \quad x'_{2j} = \frac{1}{2}\left(x_j + \frac{1}{\kappa_j}v_j\right), \quad j = 1, \ldots, n.$$

The inverse transform is given by

$$x_j = x'_{2j} + x'_{2j-1}, \quad v_j = \kappa_j(x'_{2j} - x'_{2j-1}), \quad j = 1, \ldots, n.$$

The functions X_j from the Theorem 6.1 are given by ($l = 2n$)

$$X_{2j}(x_{(2n)}) = -X_{2j-1}(x_{(2n)}) = \frac{1}{\sqrt{2}\kappa_j}F_j(x_2 + x_1, \ldots, x_{2n} + x_{2n-1}).$$

As a result we can apply the above two theorems and prove Theorem 1.1. For this it is more convenient to have another numeration of variables:

$$(x_1, x_2, x_3, \ldots, x_{2n}) \to (x_1, x_3, \ldots, x_{2n-1}, x_2, x_4, \ldots, x_{2n}).$$

In such a way (6.4) is mapped into (6.1) with

$$\lambda_j = -\sqrt{-\sigma_j}, \quad j = 1, \ldots, n, \quad \lambda_j = \sqrt{-\sigma_j}, \quad j = n+1, \ldots, 2n. \tag{6.4$'$}$$

We have also $\|x'\|_\lambda < \infty$ iff $\|x\|_\lambda < \infty$ and $\|v\|_\lambda < \infty$. Hence the Theorem 1.1 is true.

Remark that if one of σ_j is zero then it is impossible to transform the Newton equation (1.1) into the standard form (6.1).

(6.2) in the Hamiltonian form is rewritten as follows:

$$\dot{x_j} = \frac{\partial H'}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H'}{\partial x_j}, \quad H' = \sum_{j=1}^{n}\frac{p_j^2}{2m_j} + U'(x_{(n)}), \tag{6.5}$$

where the dot over the variables means the time derivative and $m_j = m$. The equation (6.5) is solved with the help of the Weinstein theorem and Lyapunov center theorem. The former is formulated as follows [9] (in [10] one can find a proof of its generalization to a case of an ordinary differential equation with an integral).

Theorem 6.3. *Let a Hamiltonian H in $\mathbb{R}^{2n}$ be twice differentiable and its Hessian determine a positive definite quadratic form at a neighborhood of the origin which is an equilibrium. If $H(0)$ is its value at the origin then for a sufficiently small $h > 0$ the Hamiltonian level $E_h = H(0) + h$ contains at least n periodic solutions of the equation of motion*

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial x_j} \tag{6.6}$$

whose periods are close to those of the linearized equation of motion.

Theorem 6.3 can be applied to (6.5) and it yields the Theorem 1.2 since the Hessian of H' determines a positive definite quadratic form in a neighborhood of the origin if the Hessian of U' has the same property. As a result the Theorem 1.2 is true for systems in which an equilibrium determines a nondegenerate minimum.

The eigenvalues of the linear part of the Hamiltonian in (6.5) are given by (6.4′). If $\sigma_j > 0$, $j = 1, \ldots, p$, then $\lambda_j = -\sqrt{-\sigma_j}$, $j = 1, \ldots, p$, are imaginary numbers and (1.1) possesses also periodic solutions due to the Lyapunov center theorem formulated as follows.

Theorem 6.4. *Let an n-dimensional Hamiltonian system have a holomorphic at the origin real-valued Hamiltonian whose Taylor power expansion begins from quadratic terms. Let also $\lambda_1, \ldots, \lambda_n$, $-\lambda_1, \ldots, -\lambda_n$ be different nonzero eigenvalues of the linear part of the Hamiltonian vector field such that the simple nonresonance condition hold for all purely imaginary eigenvalues $\lambda_j\colon \lambda_j \neq n'\lambda_s$, $j \neq s = 1, \ldots, k$, for all integers n'. Then* (6.6) *possesses k periodic solutions such that each of them depends on a real parameter c_j for some $j = 1, \ldots, k$. These solutions and their periods $\tau_1(c_1), \ldots, \tau_k(c_k)$ are holomorphic functions in the parameters at the origin and $\tau_j(0) = \dfrac{2\pi}{|\lambda_j|}$.*

Different proofs of this theorem and its generalizations can be found in [11 – 14]. There is a possibility that (1.1) admits simultaneously both periodic and bounded solutions. It is guaranteed by the following Moser theorem.

Theorem 6.5. *Let an n-dimensional Hamiltonian system have a holomorphic at the origin real-valued Hamiltonian, whose Taylor power expansion begins from quadratic terms and $\lambda_1, \ldots, \lambda_n$, $-\lambda_1, \ldots, -\lambda_n$ be eigenvalues of the linear part of the Hamiltonian vector field in $\mathbb{R}^{2n}$ which are all different nonzero complex numbers. Let also λ_1, λ_2 be independent over the reals and the following nonresonance condition $\lambda_j \neq n_1\lambda_1 + n_2\lambda_2$, $j \geq 3$, hold for all integers n_1, n_2. Then there exists a four parameter family of solutions of* (6.6) *of the form*

$$x_j(t) = \varphi_j(\xi_1, \xi_2, \eta_1, \eta_2), \quad p_j(t) = \psi_j(\xi_1, \xi_2, \eta_1, \eta_2), \tag{6.7}$$

where

$$\xi_j = e^{ta_j}\xi_j^0, \quad \eta_j = e^{-ta_j}\eta_j^0, \quad j = 1, 2,$$

and φ_j, ψ_j, a_j are holomorphic functions in a neighborhood of the origin, the latter of which depend on the complex $\xi_1^0\eta_1^0$, $\xi_2^0\eta_2^0$ and have the constant terms coinciding with λ_j. If λ_1, λ_2, $-\lambda_1$, $-\lambda_2$ contain their complex conjugates then the solution can be chosen to be real, depending on four real parameters. Moreover the matrix

$$\begin{pmatrix} \dfrac{\partial \varphi_j}{\partial \xi_s} & \dfrac{\partial \varphi_j}{\partial \eta_s} \\[2mm] \dfrac{\partial \psi_j}{\partial \xi_s} & \dfrac{\partial \psi_j}{\partial \eta_s} \end{pmatrix}$$

has rank four and if λ_1 or λ_2 is pure imaginary then the same is true for a_1 or a_2.

If λ_1 or λ_2 is positive then one has to put $\xi_1^0 = 0$ or $\xi_2^0 = 0$ in order to have the solutions on the infinite time interval. This theorem finds application in the three body problem in the celestial mechanics.

All the formulated theorems except the Weinstein theorem are constructive: the solutions of the equation of motion are expressed in terms of convergent series. The Weinstein theorem is a generalization of the Berger theorem which demands smoothness of a potential energy [13].

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