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ON REMOVABLE SETS FOR DEGENERATED ELLIPTIC EQUATIONS

ПРО МНОЖИНИ, ЩО УСУВАЮТЬСЯ, ДЛЯ ВИРОДЖЕНИХ ЕЛІПТИЧНИХ РІВНЯНЬ

We establish the necessary and sufficient conditions of compact removability.

Встановлено необхідні та достатні умови компактної усувності.

1. Introduction. The questions of compact removability for Laplace equation is studied by Carleson [1]. The uniform elliptic equation of the seconds order of divergent structure is studied by E. I. Moiseev [2]. The compact removability for elliptic and parabolic equations of nondivergent structure is considered by E. M. Landis [3]. T. S. Gadjiev, V. A. Mamedova [4]. The removability condition of compact in the space of continuous functions are constructed in the papers Harvey, Polking [5], T. Kilpelainen [6]. The different questions of qualitative properties of solutions of uniformly degenerated elliptic equations is studied by S. Chanillo, R. Z. Wreeden [7]. Uniform elliptic operator of the second order of divergent structure is considered in the paper [8].

Let E_n be n dimensional Euclidean space of the points $x=(x_1,\ldots,x_n)$. Denote by $R>0$ for $B_R\left(x_R^0\right)$ the ball $\left\{x\colon \left|x-x^0\right|<R\right\}$, and by $Q_T^R\left(x_R^0\right)$ the cylinder $B_R\left(x^0\right)\cup(0,T)$. Further let for $x^0\in E_n$, $R>0$ and $k>0$ $\varepsilon_{r,k}\left(x^0\right)$ be an ellipsoid $\left\{x\colon \sum_{i=1}^n\dfrac{\left(x_i-x_i^0\right)^2}{R^{\alpha_i}}<(kR)^2\right\}$. Let D be an bounded domain E_n with the domain ∂D, $0\in D$. ε is a such king of ellipsoid that $\overline{D}\subset\varepsilon$, $\mathfrak{B}(\varepsilon)$ is a set of all functions, satisfying in $\overline{\varepsilon}$ the uniform Lipschitz condition and having zero near the $\partial\varepsilon$.

Denote by α and $(\alpha_1,\ldots,\alpha_n)$ the vector $\langle\alpha\rangle=\alpha_1,\ldots,\alpha_n$.

Denote by $W_{2,\alpha}^1(D)$ the Banach space of the functions $u(x)$ given on D with the finite norm

$$\|u\|_{W_{2,\alpha}^1(D)}=\left(\int\limits_D\left(u^2+\sum_{i=1}^n\lambda_i(x)u_i^2\right)dx\right)^{1/2},$$

where

$$u_i=\frac{\partial u}{\partial x_i},\quad i=1,\ldots,n,\qquad \lambda_i(x)=\left(|x|_\lambda\right)^{\alpha_i},\qquad |x|_\alpha=\sum_{i=1}^n|x_i|\,\frac{2}{2+\alpha_i},$$
$$0\le\alpha_i<\frac{2}{n-1}. \tag{1}$$

Further, let $\overset{\circ}{W}{}_{2,\alpha}^1(D)$ be a degenerated set of all functions from $C_0^\infty(D)$ by the norm of the space $W_{2,\alpha}^1(D)$. Denote by $\mathcal{M}(D)$ the set of all bounded in D functions.

Let $E\subset D$ be some compact. Denote by $A_E(D)$ the totality of all functions $u(x)\in C^\infty\left(\overline{D}\right)$, each of which there exists some neighbourhood of the compact E, in which $u(x)=0$.

The compact E is called the removable relative to the first boundary-value problem for the operator L in the space $\mathcal{M}(D)$, if all generalized solution of the equation $\mathcal{L}u = 0$ in $\partial\,/E$ formed in zero on ∂D and belonging to the space $\mathcal{M}(D)$, identically equal to zero. We'll say that the function $u(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ is nonnegative on the set $H \subset \varepsilon$, in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, if there exists the sequence of the functions $\left\{u_{(m)}(x)\right\}$, $m = 1, 2, \ldots$, such that $u_m(x) \in \mathfrak{B}(\varepsilon)$, $u_m(x) \geq 0$ for $x \in H$ and $\lim_{m\to\infty} \left\|u_{(m)} - u\right\|_{W^1_{2,\alpha}(\varepsilon)} = 0$.

The function $u(x) \in W^1_{2,\alpha}(D)$ is nonnegative and ∂D in sense $W^1_{2,\alpha}(D)$, if there exists the sequence of the functions $\{u_m(x)\}$, $m = 1, 2, \ldots$, such, that $u_{(m)}(x) \in C^1(D)$, $u_m(x) \geq 0$ for $x \in \partial D$ and $\lim_{m\to\infty} \left\|u_{(m)} - u\right\|_{W^1_{2,\alpha}(\varepsilon)} = 0$. It is easy to determine the inequalities $u(x) \geq \mathrm{const}$, $u(x) \geq v(x)$, $u(x) \leq 0$, and also equality $u(x) = 1$ on the set H in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, if at the same time $u(x) \geq 1$ and $u(x) \leq 1$ on H, in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$.

Let $\omega(x)$ be measurable function in D, finite and positive for a.e. $x \in D$. Denote by $\mathcal{L}_{p,\omega}(D)$ the Banach space of the functions given on D, with the norm

$$\|u\|_{\mathcal{L}_{p,\omega}(D)} = \left(\int\limits_D (\omega(x))^{p/2} |u|^p\, dx\right)^{1/p}, \quad 1 < p < \infty.$$

Let $W^1_{p,\alpha}(D)$ be a Banach space of the functions given on $u(x)$, with the finite norm D:

$$\|u\|_{W^1_{p,\alpha}(D)} = \left(\int\limits_D \left(|u|^p + \sum_{i=1}^n (\lambda_i(x))^{p/2} |u_i|^p\right) dx\right)^{1/p}, \quad 1 < p < \infty.$$

Analogously to $\overset{\circ}{W}{}^1_{2,\alpha}(D)$, it is introduced the subspace $\overset{\circ}{W}{}^1_{p,\alpha}(D)$ for $1 < p < \infty$. The space, conjugated to $\overset{\circ}{W}{}^1_{p,\alpha}(D)$ we'll denote by $\overset{*}{W}{}^1_{p,\alpha}(D)$.

We'll consider the elliptic operator in the bounded domain $D \subset E_n$

$$\mathcal{L} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right).$$

In assumption, that $\|a_{ij}(x)\|$ is a real symmetric matrix with measurable in D elements, moreover for all $\xi \in E_n$ and a.e. $x \in D$ the condition

$$\gamma\sum_{i=1}^n \lambda_i(x)\xi_i^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \gamma^{-1}\sum_{i=1}^n \lambda_i(x)\xi_i^2. \tag{2}$$

Here $\gamma \in (0, 1]$ is a constant.

The function $u(x) \in W^1_{2,\alpha}(D)$ is called the generalized solution of the equation $\mathcal{L}u = f(x)$ in D, if for any function $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(D)$ the integral identity

$$\int\limits_D \sum_{i,j=1}^n a_{ij}(x)u_{x_i}\eta_{x_j}dx = \int\limits_D f\eta dx \tag{3}$$

be fulfilled. Here $f(x)$ is a given function from $\mathcal{L}_2(D)$.

Let $E \subset D$ be some compact. The function $u(x) \in W^1_{2,\alpha}(D \setminus E)$ is called generalized solution of the equation $\mathcal{L}u = f(x)$ in $D \setminus E$, vanishing on ∂D, if integral identity (3) is fulfilled for any function $\eta(x) \in A_E(D)$.

We'll assume that the coefficients of the operator $\mathcal{L}$ continued in $E_n \setminus D$ with saving condition (1), (2). For this, it is sufficient, for example, let's assume $a_{ij}(x) = \delta_{ij}\lambda_i(x)$ for $x \in E_n \setminus D$, $i, j = 1, \ldots, n$, where δ_{ij} is a Kronecker symbol.

Let $h(x) \in W^1_{2,\alpha}(D)$, $f^0(x) \in h_2(D)$, $f^i(x) \in \mathcal{L}_{2,\lambda^{-1}}(D)$, $i = 1, 2, \ldots, n$, are a given functions. Let's consider the first boundary-value problem

$$\mathcal{L}u = f^0(x) + \sum_{i=1}^{n} \frac{\partial f^i(x)}{\partial x_i}, \quad x \in D, \tag{4}$$

$$(u(x) - h(x)) \in \overset{\circ}{W}{}^1_{2,\alpha}(D). \tag{5}$$

The function $u(x) \in W^1_{2,\alpha}(D)$ we'll call generalized solution of problem (4), (5) if for any function $\eta(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(D)$ the integral identity

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} \eta_{x_j} dx = \int_D \left(-f^0 \eta + \sum_{i=1}^{n} f^i \eta_{x_i} \right) dx$$

is fulfilled.

Our aim to get the necessary and sufficient condition of compact removability E in the class of bounded functions.

2. Preliminaries statements. At first, we introduce some auxiliary statements.

Lemma 1. *If relative to the coefficients of the operator $\mathcal{L}$, condition* (1), (2) *be fulfilled, then the first boundary-value problem* (4), (5) *has a unique generalized solution $u(x)$ at any $h(x) \in W^1_{2,\alpha}(D)$, $f^0(x) \in h_2(D)$, $f^i(x) \in L_{2,\lambda_i^{-1}}(D)$, $i = 1, 2, \ldots, n$. At this there exists $P_0(\alpha, n)$ such that, if $p > p_0$, $h(x) \in W^1_{p,\alpha}(D)$, $f^0(x) \in h_p(D)$, $f^i(x) \in L_{2,\lambda_i^{-1}}(D)$, $i = 1, 2, \ldots, n$, $\partial D \in C^1$, then solution $u(x)$ is continuous in $\overline{D}$.*

Lemma 2. *Let relative to the coefficients of the operator $\mathcal{L}$ conditions* (1), (2) *be fulfilled. Then any generalized solution of the equation $\mathcal{L}u = 0$ in D is continuous by Hölder at each strictly internal domain ∂.*

Lemma 3. *Let relative to the coefficients of the operator $\mathcal{L}$, conditions* (1), (2) *be fulfilled and $\overline{\varepsilon_{R,1}} < D$. Then for any positive generalized solution $u(x)$ the equation $\mathcal{L}u = 0$ in D the Harnack inequality is true*

$$\sup_{\varepsilon_{R,1}(0)} u \leq C_1(\gamma, \alpha, n) \inf_{\varepsilon_{R,1}(0)} u. \tag{6}$$

If at this $y \in \partial\varepsilon_{R,2}(0)$ and $\overline{\varepsilon_{R,1}}(0) \subset D$, then the inequality of form (6) *is true in ellipsoid $\varepsilon_{R,1}(y)$.*

Lemma 4. *Let relative to the coefficients of the operator $\mathcal{L}$ conditions* (1), (2) *be fulfilled, and $u(x)$ be generalized solution of the first boundary-value problem* (4), (5) *at $f^i(x) \equiv 0$, $i = 0, \ldots, n$. Then if $h(x)$ is bounded on ∂D in sense $W^1_{2,\alpha}(D)$, then for solution $u(x)$ the following maximum principle is true*:

$$\inf_{\partial D} h \leq \inf_{D} u \leq \sup_{D} \leq \sup_{\partial D} h,$$

where $\inf_{\partial D} h\Big(\sup_{\partial D} h\Big)$ *is an exact lower (upper) bound those numbers* a, *for which* $h(x) \geq a$ $(h(x) \leq a)$ *on* ∂D *in sense* $W^1_{2,\alpha}(D)$.

These lemmas are proved analogously to paper [7]. Therefore, we don't give the proof of these lemmas.

Let $H \subset \varepsilon$ be some compact, V_H be a set of all functions $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, such that $\varphi(x) \geq 1$ on H, in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. Let's consider the functional

$$J_\theta(\varphi) = \int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x)\varphi_i\varphi_j dx, \quad \varphi(x) \in V_H,$$

$\mathcal{L}$ is a H compact capacity relative to ellipsoid ε is called the value $\inf_{\varphi \in V_H} J_\theta(u)$ and denoted by $\operatorname{cap}^{(\varepsilon)}_{\mathcal{L}}(H)$. In case $\varepsilon = E_n$, the corresponding value is called $\mathcal{L}$ capacity of the compact H and denoted by $\operatorname{cap}_{\mathcal{L}}(H)$.

Lemma 5. *There exists the unique function* $u(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ *such that* $u(x) \geq 1$ *on* H *in sense* $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ *and* $\operatorname{cap}^{(\varepsilon)}_{\mathcal{L}}(H) = J_{\mathcal{L}}(u)$. *Moreover,* $u(x) = 1$ *on* H *in sense* $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$.

Proof. It is easy to see that V_H is convex closed set in $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. From the fact that $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ is a Hilbert space, it follows the existence of unique function $u(x) \in V_H$, which achieved an exact lower bound of the functional $J_{\mathcal{L}}(\varphi)$. Let's next $\{u(x)\}^1 = \begin{cases} u(x) & \text{if} \quad u(x) \leq 1, \\ 1 & \text{if} \quad u(x) > 1. \end{cases}$

It is clear, that $\{u(x)\}^1 \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. Moreover, $\{u(x)\}^1 \in V_H$. Denote by $A^+ = \big\{x \colon x \in \in \varepsilon,\ u(x) > 1\big\}$. We have

$$J_{\mathcal{L}}\left\{u(x)^1\right\} = \left(\int\limits_{A^+} + \int\limits_{\varepsilon\backslash A^+}\right) \sum_{i,j=1}^{n} a_{ij}(x)\left\{u\right\}^1_i\left\{u\right\}^1_j dx = \int\limits_{\varepsilon\backslash A^+} \sum_{i,j=1}^{n} a_{ij}(x)u_iu_j dx. \tag{7}$$

On the other side, according to (1)

$$\int\limits_{A^+} \sum_{i,j=1}^{n} a_{ij}(x)u_iu_j dx \geq 0. \tag{8}$$

From (7) and (8) we conclude

$$J_{\mathcal{L}}\left\{u(x)^1\right\} \leq J_{\mathcal{L}}(u) = \inf_{\varphi \in V_H} J_{\mathcal{L}}(\varphi),$$

i.e., $J_{\mathcal{L}}\left\{u(x)^1\right\} = J_{\mathcal{L}}(u)$. From uniqueness extreme function it follows, that $\{u(x)\}^1 = u(x)$.

Lemma 5 is proved.

The function $u(x)$, which achieved an exact lower bound of the functional $J_{\mathcal{L}}(\varphi)$ on the set V_H is called $\mathcal{L}$ capacity of the compact potential H relative to the ellipsoid ε.

Lemma 6. *$\mathcal{L}$ be a capacity potential $u(x)$ of the compact H relative to ε is a generalized solution of the equation $\mathcal{L}u = 0$ in $\varepsilon\backslash H$, vanishing on 0 and $\partial\varepsilon$ in 1 on ∂H sense $W^1_{2,\alpha}(\varepsilon)$.*

Proof. It is sufficient to show the truthiness of the first part of assertion of lemma. Let $\eta(x) \in \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ and $\eta(x) \geq 0$ on H in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$. Then for any $\varepsilon > 0$ $(u(x) + \varepsilon\eta(x)) \in V_H$. Therefore

$$J_{\mathcal{L}}\left(u + \varepsilon\eta\right) \geq J_{\mathcal{L}}\left(u\right).$$

Thus

$$J_{\mathcal{L}}\left(u\right) + \varepsilon^2 J_{\mathcal{L}}\left(\eta\right) + 2\varepsilon \int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \eta_j dx \geq J_{\mathcal{L}}\left(u\right),$$

i.e.,

$$J_{\mathcal{L}}\left(u\right) + 2\varepsilon \int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \eta_j dx \geq 0.$$

Tending ε to zero, we conclude

$$\int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \eta_j dx \geq 0. \tag{9}$$

It is easy to see as $\eta(x)$ in (9) we can take any function from $C^1\left(\overline{\varepsilon}\right)$ with compact support in $\varepsilon\backslash H$. Then

$$\int\limits_{\varepsilon\backslash H} \sum_{i,j=1}^{n} a_{ij}(x) u_i \eta_j dx \geq 0.$$

Substituting $\eta(x)$ on $-\eta(x)$, we arrive to the equality

$$\int\limits_{\varepsilon\backslash H} \sum_{i,j=1}^{n} a_{ij}(x) u_i \eta_j dx = 0.$$

Lemma 6 is proved.

Let μ be a charge of bounded variation, given on ε. We'll say, that the function $u(x) \in L_1(\varepsilon)$ is a weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$, if for any function $\varphi(x) \in \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon) \operatorname{cap} C\left(\overline{\varepsilon}\right)$ the integral identity

$$\int\limits_{\varepsilon} u\mathcal{L}\varphi dx = \int\limits_{\varepsilon} \varphi d\mu$$

is fulfilled.

According to Lemma 1 (at $h = 0$) there exists the continuous linear operator H from $\overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$ in $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, such that for any functional $T \in \overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$, the function $u = H\left(T\right)$ is unique in $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ generalized solution of the equation $\mathcal{L}u = T$.

The operator H is called Green operator.

By Lemma 1 this operator at $p > p_0$ we transform $\overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$ to $C\left(\overline{\varepsilon}\right)$. It is easy to see, that the function $u(x)$ is weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$, iff for any function $\psi(x) \in C\left(\overline{\varepsilon}\right)$ the integral identity

$$\int\limits_{\varepsilon} u\psi dx = \int\limits_{\varepsilon} H(\psi) d\mu \tag{10}$$

is fulfilled.

By analogy with [8] we can show that for each measure μ on ε there exists the unique weak solution of the equation $\mathcal{L}u = -\mu$ equaling to zero on $\partial\varepsilon$.

Let's say, that the charge $\mu \in \overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$ if there exists the vector $\overline{f}(x) = \big(f^0(x), f^1(x), \ldots, f^n(x)\big)$, $f^0(x) \in h_2(\varepsilon)$, $f^i(x) \in L_{2,\lambda_i}(\varepsilon)$, $i = 1, 2, \ldots, n$, for any function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon) \operatorname{cap} C(\overline{\varepsilon})$ the integral identity

$$\mu(\varphi) = \int\limits_{\varepsilon} \varphi d\mu = \int\limits_{\varepsilon} \left(f^0\varphi - \sum_{i=1}^{n} f^i \varphi_i \right) dx$$

is true.

At this, it is evident that

$$\left| \int\limits_{\varepsilon} \varphi d\mu \right| \le C_2\left(\overline{f}\right) \|\varphi\|_{W^1_{2,\alpha}(\varepsilon)}.$$

Lemma 7. *The weak solution $u(x)$ of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$, belongs to $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, iff $\mu \in \overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$.*

Proof. At first, we'll show that if the function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ satisfies the integral identity

$$\int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx = -\int\limits_{\varepsilon} \varphi d\mu \tag{11}$$

for any function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon) \operatorname{cap} C(\overline{\varepsilon})$, then it is weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$. Really, assuming $\varphi = H(\psi)$, $\psi(x) \in C(\overline{\varepsilon})$ we obtain

$$\int\limits_{\varepsilon} H(\psi) d\mu = \int\limits_{\varepsilon} \varphi d\mu = -\int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx =$$

$$= \int\limits_{\varepsilon} u \sum_{i,j=1}^{n} \left(a_{ij}(x)\varphi_j\right)_i dx = \int\limits_{\varepsilon} u \mathcal{L}\varphi dx = \int\limits_{\varepsilon} u\psi dx,$$

and now it is sufficient to use the identity (10). We'll show that $\mu \in \overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$. For this, it is sufficient to prove, that if $f^i(x) = \sum_{i=1}^{n} a_{ij}(x) u_i(x)$, then $f^i(x) \in L_{2,\lambda_i^{-1}}(\varepsilon)$, $i = 1, 2, \ldots, n$. Assume in condition (11) $\xi_1 = \ldots = \xi_{i-1} = \xi_{i+1} = \ldots = \xi_n = 0$, $\xi_i = \dfrac{1}{\sqrt{\lambda_i(x)}}$.

We have

$$\gamma \le \frac{a_{ii}(x)}{\lambda_i(x)} \le \gamma^{-1}, \quad i = 1, \ldots, n. \tag{12}$$

Let $i \ne j$. Assuming $\xi_k = 0$ at $k \ne j$ and $k \ne i$, $\xi_i = \dfrac{1}{\sqrt{\lambda_i(x)}}$, $\xi_j = \dfrac{1}{\sqrt{\lambda_j(x)}}$, we get

$$2\gamma \le \frac{a_{ii}(x)}{\lambda_i(x)} + \frac{a_{jj}(x)}{\lambda_j(x)} + \frac{2a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}} \le 2\gamma^{-1}.$$

Using (12), we conclude

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \le \gamma^{-1} - \gamma, \qquad i,j = 1,\ldots,n, \quad i \ne j. \tag{13}$$

From (12) and (13) it follows that

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \le \gamma^{-1}, \quad i,j = 1,\ldots,n. \tag{14}$$

Thus, from (14) take out for $j = 1,\ldots,n$

$$\int\limits_{\varepsilon} \frac{1}{\lambda_j(x)} \left(f^j\right)^2 dx = \int\limits_{\varepsilon} \frac{1}{\lambda_j(x)} \left(\sum_{i=1}^{n} a_{ij}(x) u_i\right)^2 dx \le \gamma^{-2} n \sum_{i=1}^{n} \int\limits_{\varepsilon} \lambda_i(x) u_i^2 dx < \alpha.$$

So, $\mu \in \overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$. Inversely, if $u(x)$ is a weak solution of the equation $\mathcal{L}u = -\mu$, vanishing on $\partial\varepsilon$, then there exists $\mu \in \overset{*}{W}{}^1_{2,\alpha}(\varepsilon)$, such that

$$\left(f^0\varphi - \sum_{i=1}^{n} f^i \varphi_i\right) dx = \int\limits_{\varepsilon} \varphi d\mu = \int\limits_{\varepsilon} u \mathcal{L}\varphi dx =$$

$$= \int\limits_{\varepsilon} u \sum_{i,j=1}^{n} \left(a_{ij}(x)\varphi_j\right)_i dx = -\int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j dx$$

for any function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon) \operatorname{cap} C\left(\overline{\varepsilon}\right)$, $\mathcal{L}\varphi(x) \in C\left(\overline{\varepsilon}\right)$.

Then, from Lemma 1 we obtain that $u(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$.

Lemma 7 is proved.

Let now $\delta(x)$ be Dirac measure, concentrated at the point 0, y is an arbitrary fixed point ε.

The weak solution $g(x,y)$ of the equation $\mathcal{L}y = -\delta(x-y)$, vanishing on $\partial\varepsilon$ is called Green function of the operator $\mathcal{L}$ in ε.

In case $\varepsilon = E_n$ the corresponding function is called the fundamental solution of the operator $\mathcal{L}$ and denoted by $G(x,y)$.

According to above proved, if $\psi(x)$ is an arbitrary function from $C\left(\overline{\varepsilon}\right)$, then the generalized solution $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ of the equation $\mathcal{L}\varphi = -\psi$ can be introduced in the following from:

$$\varphi(y) = \int\limits_{\varepsilon} g(x,y)\psi(x)dx.$$

We can show, that $g(x,y)$ is nonnegative in $\varepsilon \times \varepsilon$, moreover, $g(x,y) = g(y,x)$.

Lemma 8. *For any charge, of bounded variation on ε the integral*

$$u(x) = \int\limits_{\varepsilon} g(x,y) d\mu(y)$$

exists, finite a.e. in ε and is weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$.

Proof. Without losing generality, we'll assume that the charge μ is the measure in ε. Let $\varphi(x) \in C(\overline{\varepsilon})$, $\psi(x) \geq 0$ in ε. Denote by $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$ the generalized solution of the equation $\mathcal{L}\varphi = -\psi(x)$. Then $\varphi(x) \in C(\overline{\varepsilon})$ according to Lemma 1 and $\psi(x) \geq 0$ according to Lemma 4. At this

$$\varphi(y) = \int\limits_{\varepsilon} g(x,y)\psi(x)dx.$$

Then, by Fubini theorem we conclude, that the integral $\int\limits_{\varepsilon} g(x,y)d\mu(y)$ there exists for almost all $x \in \varepsilon$, moreover

$$\int\limits_{\varepsilon} H(\psi)d\mu(y) = \int\limits_{\varepsilon} \varphi(y)d\mu(y) = \iint\limits_{\varepsilon\times\varepsilon} g(x,y)\psi(x)dxd\mu(y) = \int\limits_{\varepsilon} \psi(x)u(x)dx. \tag{15}$$

Let's note, that the equality (15) is fulfilled for weak nonnegative and continuous in $\overline{\varepsilon}$ function $\psi(x)$. Now, it is sufficient to remember the identity (10).

Lemma 8 is proved.

Let's consider now $\mathcal{L}$-capacity of the potential $u(x)$ of the compact H relative to the ellipsoid ε. Before, it was proved that $u(x)$ satisfies the inequality (9) at any nonnegative on H the function $\eta(x) \in C_0^\infty(\varepsilon)$. By the Schwartz theorem [9] there exists the measure μ on H such that

$$\int\limits_{\varepsilon} \sum_{i,j=1}^{n} a_{ij}(x)u_i\eta_j dx = \int\limits_{\varepsilon} \eta d\mu. \tag{16}$$

Further, since $u = 1$ on H in sense $\overset{\circ}{W}{}^1_{2,\alpha}(\varepsilon)$, then the carrier of the measure μ is situated on ∂H. The measure μ is called $\mathcal{L}$ -capacity distribution of the compact H.

According to Lemma 8 $\mathcal{L}$-capacity potential $u(x)$ is weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$ and can be represented in the following form:

$$u(x) = \int\limits_{\varepsilon} g(x,z)\,d\mu(z). \tag{17}$$

On the other side, there exists the sequence of the functions $\left\{\eta^{(m)}(x)\right\}$, $m = 1, 2, \ldots$, such that $\eta^{(m)}(x) \in \mathfrak{B}(\varepsilon)$, $\eta^{(m)}(x) = 1$ for $x \in H$ and $\lim_{m\to\infty} \left\|\eta^{(m)} - u\right\|_{W^1_{2,\alpha}(\varepsilon)} = 0$. Assuming in equality (10) $\eta^{(m)}(x)$ instead of $\eta^{(m)}$, we conclude that it first fart is equal to $\mu(H)$ at any natural m, while the left part tends to $\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}(H)$ as $m \to \infty$. Thus,

$$\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}(H) = \mu(H). \tag{18}$$

Lemma 9. *Let relative to coefficients of the operator $\mathcal{L}$ conditions* (1), (2), $y \in \partial\varepsilon_{R,2}(0)$, $\overline{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial\varepsilon_{R,1}(y)$ *be fulfilled. Then for the Green function $g(x,y)$ the following estimations are true*:

$$C_3(\gamma,\alpha,n)\left[\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\left(\overline{\varepsilon}_{R,1}(y)\right)\right]^{-1} \leq g(x,y) \leq C_4(\gamma,\alpha,n)\left[\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\left(\overline{\varepsilon}_{R,1}(y)\right)\right]^{-1}. \tag{19}$$

If $\overline{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial\varepsilon_{R,1}(0)$, *then*

$$C_3\left[\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\left(\overline{\varepsilon}_{R,1}(0)\right)\right]^{-1} \leq g(x,0) \leq C_4\left[\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\left(\overline{\varepsilon}_{R,1}(0)\right)\right]^{-1}. \tag{20}$$

Proof. Without loss of generality, we can assume that the coefficients of the operator $\mathcal{L}$ are continuously differentiable in $\overline{\varepsilon}$. The general case is obtained by means of limit passage. Then at $x \neq y$ the function $g(x, y)$ is continuous by x and y, moreover

$$\lim_{x \to y} g(x, y) = \infty. \tag{21}$$

Let a be a positive number, which will be chosen later, $K_a = \{x \colon g(x, y) \geq a\}$, where y is an arbitrary fixed point on $\partial\varepsilon_{R,2}(0)$. From (21) it follows that y is internal point y of the compact K_a. Then $\mathcal{L}$ is capacity potential K_a, represented in the form (17), so it means it equal to zero in it. Thus,

$$1 = \int\limits_{\varepsilon} y\,(y, z)\, d\mu_a(z),$$

where μ is a $\mathcal{L}$-capacity distribution of the compact K_a. Allowing for the carrier of the measure μ_a is situated on ∂K_a, where $g\,(y, z) = a$ and using (18) , we obtain

$$\mu_a\,(K_a) = \operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\,(K_a) = \frac{1}{a}. \tag{22}$$

Let's assume now, $a = \inf_{x \in \partial\varepsilon_{R,1}(y)} g(x, y)$. According to maximum principle $\overline{\varepsilon}_{R,1}(y) \subset K_a$. Therefore from (22) we conclude

$$\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\,(\overline{\varepsilon}_{R,1}(y)) \leq \operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\,(K_a) = \frac{1}{\inf\limits_{x \in \partial\varepsilon_{R,1}(y)} g(x, y)}. \tag{23}$$

If we'll assume $b = \sup_{x \in \partial\varepsilon_{R,1}(y)} g(x, y)$, then $\overline{\varepsilon}_{R,1}(y) \subset K_a$, i.e.,

$$\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\,(\overline{\varepsilon}_{R,1}(y)) \leq \operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\,(K_b) = \frac{1}{\sup\limits_{x \in \partial\varepsilon_{R,1}(y)} g(x, y)}. \tag{24}$$

From (23) and (24) follows that

$$\inf_{x \in \partial\varepsilon_{R,1}(y)} g(x, y) \leq \left[\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\,(\overline{\varepsilon}_{R,1}(y))\right]^{-1} \leq \sup_{x \in \partial\varepsilon_{R,1}(y)} g(x, y). \tag{25}$$

On the other side, according to Lemma 3

$$\sup_{x \in \partial\varepsilon_{R,1}(y)} g(x, y) \leq C_5\,(\gamma, \alpha, n) \inf_{x \in \partial\varepsilon_{R,1}(y)} g(x, y). \tag{26}$$

Now, the required estimations (19) follows from (25) and (26). Absolutely analogously the truthiness of equalities (20) is proved.

***Corollary* 1.** *Let the conditions of the lemma, and $y \in \partial\varepsilon_{R,2}(0)$ be fulfilled, $\overline{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial\varepsilon_{R,1}(0)$ or $y = 0$, $\overline{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial\varepsilon_{R,1}(0)$. Then for fundamental solution $G(x, y)$ the estimations*

$$C_3 \left[\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\,(\overline{\varepsilon}_{R,1}(0))\right]^{-1} \leq G(x, y) \leq C_4 \left[\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\,(\overline{\varepsilon}_{R,1}(0))\right]^{-1} \tag{27}$$

are true.

3. Removability criterion of the compact in the space $M(D)$.

Theorem 1. *Let relative to the coefficients of the operator $\mathcal{L}$, conditions* (1), (2) *be fulfilled. Then for removability of the compact $E \subset D$ relative to the first boundary-value problem for the operator $\mathcal{L}$ in the space $\mathcal{M}(D)$ it is necessary and sufficient, that*

$$\operatorname{cap}_{\mathcal{L}}(E) = 0. \tag{28}$$

Proof. Let the ellipsoid ε has the same sense, that above. It is easy to see that if condition (28) be fulfilled, then

$$\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}(E) = 0.$$

Not losing generality, we can limited with case, when the coefficients of the operator $\mathcal{L}$ is continuously differentiable in $\overline{\varepsilon}$. Let's fixed an arbitrary $\varepsilon > 0$ and $x^0 \subset D \backslash E$. By virtue of (28) there exists the neighbourhood H of the compact E, such that

$$\operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\left(\overline{H}\right) < \varepsilon. \tag{29}$$

At this, we can assume that ε is such small, that

$$\operatorname{dist}\left(x^0, \overline{H}\,\right) \geq \frac{1}{2} \operatorname{dist}\left(x^0, E\right). \tag{30}$$

Denote by $V_H(x)$ and μ_H the $\mathcal{L}$-capacity potential of the compact $\overline{H}$ relative to the ellipsoid ε and $\overline{\mathcal{L}}$-capacity of the distribution $\overline{H}$, respectively. According to above proved

$$V_H(x) = \int\limits_{\varepsilon} g(x,y) d\mu_H(y),$$

moreover the function $V_H(x)$ is generalized solution of the equation $\mathcal{L}V_H = 0$ in $\varepsilon \backslash \overline{H}$, vanishing on 0 and in $\partial\varepsilon$ on 1 in ∂H sense $W^1_{2,\alpha}(\varepsilon)$. Let now, $u(x) \in \mathcal{M}(D)$ is an arbitrary solution of the equation $\mathcal{L}u = 0$ in $D \backslash E$, vanishing on ∂D, $M = \sup_D |u|$. It is easy to see, that the function $V_H(x)$ is nonnegative on ∂D, in sense $W^1_{2,\alpha}(D)$. Hence, it follows, that the function $u(x) - MV_H(x)$ is generalized solution of the equation $\mathcal{L}u = 0$ in $D \backslash \overline{H}$, is nonpositive on $\partial\left(D \backslash \overline{H}\right)$. According to Lemma 4 $u(x) - MV_H(x) \leq 0$ and $D \backslash \overline{H}$ in particular

$$u\left(x^0\right) \leq MV_H\left(x^0\right) \leq M \sup_{y \in \partial H} g\left(x^0, y\right) \mu_H\left(\overline{H}\right) = M \sup_{y \in \partial H} g\left(x^0, y\right) \operatorname{cap}_{\mathcal{L}}^{(\varepsilon)}\left(\overline{H}\right). \tag{31}$$

By virtue of continuity of the function $g(x,y)$ at $x \neq y$ and inequality (30) we obtain

$$\sup_{y \in \partial H} g\left(x^0, y\right) \leq C_6\left(\gamma, \alpha, n, x^0, E\right).$$

Thus, from (29) and (31) we conclude

$$u\left(x^0\right) \leq MC_6\varepsilon. \tag{32}$$

Using an arbitrariness ε, we lead to the inequality

$$u\left(x^0\right) \leq 0. \tag{33}$$

Making analogous considerations with the function $u(x) + MV_H(x)$, we have

$$u\left(x^0\right) \geq 0. \tag{34}$$

From (32), (33) and an arbitrariness of the point x^0 it follows, that $u(x) \equiv 0$ in $D\backslash E$. Thereby, the sufficiency of condition (28) is proved. Let's prove its necessarily. Let's assume that $\operatorname{cap}_{\mathcal{L}}(E) > 0$. Denote by ε' the ellipsoid, such that $\overline{\varepsilon}' \subset \delta$, $E \subset \varepsilon'$. Assume $D = \varepsilon$. Further, let $u_E(x)$ be V_E-$\mathcal{L}$ capacity potential of the compact E relative to the ellipsoid ε' and $\mathcal{L}$-capacity distribution E, respectively. Following to [10], we can give the equivalent definition of Vallee Poussin type of $\mathcal{L}$-capacity of the compact E, relative to the ellipsoid ε'. Let $g(x,y)$ be a Green function of the operator $\mathcal{L}$ in ε'. Let's call the measure μ on E, $\mathcal{L}$-admissible, if $\mu \subset E$ and

$$V_\mu^E(x) = \int\limits_{\varepsilon'} g(x,y)d\mu(y) \leq 1 \quad \text{for} \quad x \in \sup p\mu. \tag{35}$$

The value $\sup \mu(E) = \operatorname{cap}_{\mathcal{L}}^{(\varepsilon')}(E)$, where an exact upper boundary is taken on all $\mathcal{L}$-admissible measures, is called $\mathcal{L}$-capacity of the compact E, relative to the ellipsoid ε'.

Analogously, the $\mathcal{L}$-capacity $\operatorname{cap}_{\mathcal{L}}(E)$ is determined. At this by the standard method we show, that there exists the unique measure, on which an exact upper boundary of the functional $\mu(E)$ is reached, by the set of all $\mathcal{L}$-admissible measures μ. This measure is $\mathcal{L}$-capacity distribution of the compact E.

According to the above proved, the function $u_E(x)$ is generalized solution of the equation $\mathcal{L}u_E = 0$ in $\varepsilon' \backslash E$, equaling to zero on $\partial\varepsilon'$. Besides, from (34) and maximum principle it follows that $u_E(x) \in M\left(\varepsilon'\right)$. On the other side $u_E(x) \not\equiv 0$, as $V_H(E) > 0$.

Theorem 1 is proved.

Lemma 10. *Let relative to the coefficients of the operator $\mathcal{L}$ condition* (1) *be fulfilled. Then, if $y \in \partial\varepsilon_{R,2}(0)$, then $C_7\left(\gamma,\alpha,n\right) R^{n+\frac{\langle\alpha\rangle}{2}-2} \leq \operatorname{cap}_{\mathcal{L}}\left(\overline{\varepsilon}_{R,1}(y)\right) \leq C_8\left(\gamma,\alpha,n\right) R^{n+\frac{\langle\alpha\rangle}{2}-2}$.*

Proof. Let $\mathcal{L}_0 = \sum_{i=1}^n \dfrac{\partial}{\partial x_i}\left(\lambda_i(x)\dfrac{\partial}{\partial x_i}\right)$. Then, according to (1)

$$\gamma \operatorname{cap}_{\mathcal{L}_0}\left(\overline{\varepsilon}_{R,1}(y)\right) \leq \operatorname{cap}_{\mathcal{L}}\left(\overline{\varepsilon}_{R,1}(y)\right) \leq \gamma^{-1} \operatorname{cap}_{\mathcal{L}_0}\left(\overline{\varepsilon}_{R,1}(y)\right). \tag{36}$$

Let $u(x) \in C_0^\infty\left(\varepsilon_{R,\frac{3}{2}}(y)\right)$, $u(x) = 1$ for $\varepsilon_{R,1}(y)$, moreover

$$|u_i(x)| \leq \frac{C_9\left(\lambda,n\right)}{R^{1+\frac{\alpha_i}{2}}}, \quad i = 1,\ldots,n. \tag{37}$$

Then

$$\operatorname{cap}_{\mathcal{L}_0}\left(\overline{\varepsilon}_{R,1}(y)\right) \leq \int\limits_{\varepsilon_{R,\frac{3}{2}}(y)} \sum_{i=1}^n \lambda_i(x) u_i^2 dx. \tag{38}$$

On the other side, as $y \in \partial\varepsilon_{R,2}(0)$, then $\sum_{i=1}^n \dfrac{y_i^2}{R^{\alpha_i}} = 4R^2$ and thereby

$$|y_i| \leq 2R^{1+\frac{\alpha_i}{2}}, \quad i = 1,\ldots,n.$$

Besides, as $x \in \varepsilon_{R,\frac{3}{2}}(y)$, then

$$|x_i - y_i| \le \frac{3}{2} R^{1+\frac{\alpha_i}{2}}, \quad i = 1, \ldots, n.$$

Thus

$$|x_i| \le |y_i| + |x_i - y_i| \le \frac{7}{2} R^{1+\frac{\alpha_i}{2}}, \quad i = 1, \ldots, n.$$

Hence, it follows that

$$|x|_\alpha \le R \sum_{i=1}^{n} \left(\frac{z}{2}\right)^{\frac{2}{2+\lambda_i}}.$$

Therefore

$$\lambda_i(x) \le C_{10}^{\alpha_i} R^{\alpha_i} \le C_{10}^{\alpha^+} R^{\alpha_i}, \quad i = 1, \ldots, n. \tag{39}$$

where $\alpha^+ = \max\{\alpha_1, \ldots, \alpha_n\}$.

Allowing for (37) and (39) in (38) we obtain

$$\operatorname{cap}_{\mathcal{L}_0}(\overline{\varepsilon}_{R,1}(y)) \le C_{10}(\alpha, n) R^{-2} \operatorname{mes}\left(\varepsilon_{R,\frac{3}{2}}(y)\right) = C_{11}(\alpha, n) R^{n+\frac{\langle\alpha\rangle}{2}-2}$$

and by virtue of (36), the estimation from upper in (35) is proved.

For showing the truthiness of the estimations from lower in (35), we note that

$$\operatorname{cap}_{\mathcal{L}_0}(\overline{\varepsilon}_{R,1}(y)) \ge \operatorname{cap}_{\mathcal{L}_0}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right). \tag{40}$$

Besides, considering the same as in [8], we conclude

$$\operatorname{cap}_{\mathcal{L}_0}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right) \ge C_{12}(\alpha, n) \operatorname{cap}_{\mathcal{L}_0}^{(\varepsilon_0)}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right), \tag{41}$$

where $\varepsilon_0 = \varepsilon_{R,\frac{1}{\sqrt{n}}}(y)$.

Let $W = \left\{u(x) \colon u(x) C_0^\infty(\varepsilon_0),\ u(x) = 1 \text{ for } x \in \varepsilon_{R,\frac{1}{2\sqrt{n}}}(y)\right\}$. Then

$$\operatorname{cap}_{\mathcal{L}_0}^{(\varepsilon_0)}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right) = \inf_{u \in W} \int_{\varepsilon_0} \sum_{i=1}^{n} \lambda_i(x) u_i^2 dx. \tag{42}$$

On the other side, if $y \in \partial\varepsilon_{R,2}(0)$, then we can find i_0, $1 \le i_0 \le n$, such that $y_{i_0}^2 \ge \dfrac{4R^{2+\alpha_{i_0}}}{n}$, i.e.,

$$|y_{i_0}| \ge \frac{4R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}.$$

Besides, as $x \in \varepsilon_0$, then

$$|x_{i_0} - y_{i_0}| \le \frac{R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}.$$

Therefore

$$|x_{i_0}| \ge |y_{i_0}| - |x_{i_0} - y_{i_0}| \ge \frac{R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}.$$

Thereby

$$\lambda_i(x) \geq n^{-\frac{1}{2+\alpha_{i_0}}} R, \quad i = 1, \ldots, n, \tag{43}$$

where $\alpha^- = \min\{\alpha_1, \ldots, \alpha_n\}$.

Allowing for (43) in (42) we have

$$\operatorname{cap}_{\mathcal{L}_0}^{(\varepsilon_0)}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right) = C_{13}(\alpha, n) \inf_{u \in W} \int\limits_{\varepsilon_0} \sum_{i=1}^{n} R^{\alpha_i} u_i^2 dx. \tag{44}$$

Denote by $B_R(z)$ the ball $\{x\colon |x - z| < R\}$. Let's make in (44) the substitution of the variables $v_i = \dfrac{x_i}{R^{1+\frac{\alpha_i}{2}}}$, $i = 1, \ldots, n$, and let $\widetilde{y}$ is an image of the point y, where $\widetilde{W} = \Big\{\widetilde{u}(v)\colon$ $\widetilde{u}(\tau) C_0^\infty(B_0)$, $\widetilde{u}(\tau) = 1$ for $v \in B_{\frac{1}{2\sqrt{n}}}(\widetilde{y})\Big\}$. Then from (44) we deduce $B_0 = B_{\frac{1}{2\sqrt{n}}}(\widetilde{y})$ where by

$$\operatorname{cap}_{\mathcal{L}_0}^{(\varepsilon_0)}\left(\overline{\varepsilon}_{R,\frac{1}{2\sqrt{n}}}(y)\right) \geq C_{13} R^{n+\frac{\langle\alpha\rangle}{2}-2} \inf_{\widetilde{u} \in \widetilde{W}} \int\limits_{B_0} \sum_{i=1}^{n} \left(\frac{\partial \widetilde{u}}{\partial v_i}\right)^2 d\tau =$$

$$= C_{13} R^{n+\frac{\langle\alpha\rangle}{2}-2} \operatorname{cap}^{(B_0)}\left(\overline{B}_{\frac{1}{2\sqrt{n}}}(\widetilde{y})\right), \tag{45}$$

we'll denote by $\operatorname{cap}^{(B_0)}\left(\overline{B}_{\frac{1}{2\sqrt{n}}}(\widetilde{y})\right)$ Wiener capacity of the compact $\overline{B}_{\frac{1}{2\sqrt{n}}}(\widetilde{y})$, relative to the ball B_0. Now, it is sufficient to note that $\operatorname{cap}^{(B_0)}\left(\overline{B}_{\frac{1}{2\sqrt{n}}}(\widetilde{y})\right) = C_{14}(n)$ and required estimation follows from (40), (41) and (45).

Lemma 10 is proved.

Lemma 11. *Let relative to the coefficients of the operator $\mathcal{L}$ condition* (1) *be fulfilled. Then*

$$C_{15}(\gamma, \alpha, n) R^{n+\frac{\langle\alpha\rangle}{2}-2} \leq \operatorname{cap}_{\mathcal{L}}(\overline{\varepsilon}_{R,1}(y)) \leq C_{16}(\gamma, \alpha, n) R^{n+\frac{\langle\alpha\rangle}{2}-2}. \tag{46}$$

Upper estimation in (46) *is proved analogously to the estimation in* (35). *For the proofing of the lower estimation, it is sufficient to note that $\varepsilon_{R,\frac{1}{4}}(\overline{y}) \subset \varepsilon_{R,1}(0)$, i.e.,*

$$\operatorname{cap}_{\mathcal{L}}\left(\overline{\varepsilon}_{R,\frac{1}{4}}(\overline{y})\right) < \operatorname{cap}_{\mathcal{L}}(\overline{\varepsilon}_{R,1}(0)), \tag{47}$$

where $\overline{y} = \left(\dfrac{1}{2} R^{1+\frac{\alpha}{2}}, 0, \ldots, 0\right)$ and repeat the consideration of the proofing of the previous lemma.

Corollary 2. *If conditions* (1), (2) *$y \in \partial\varepsilon_{R,2}(0)$ be fulfilled, then for any $\rho \in (0, R]$ the estimation*

$$\operatorname{cap}_{\mathcal{L}}(\overline{\varepsilon}_{\rho,1}(\overline{y})) \leq C_{17}(\gamma, \alpha, n) \rho^{n+\frac{\langle\alpha\rangle}{2}-2} \left(1 + \sum_{i=1}^{n} \left(\frac{R}{\rho}\right)^{\alpha_i}\right) \tag{48}$$

is true.

Then $v(x) \in C_0^\infty\left(\varepsilon_{\rho,\frac{3}{2}}(y)\right)$, $v(x) = 1$ for $x \in \varepsilon_{\rho,1}(y)$

$$|v_i(x)| \leq \frac{C_{18}(\alpha, n)}{\rho^{1+\frac{\alpha_i}{2}}}, \quad i = 1, \ldots, n,$$

$$\operatorname{cap}_{\mathcal{L}_0}\left(\overline{\varepsilon}_{\rho,1}\left(\overline{y}\right)\right)=\gamma^{-1}C_{18}^{2}\rho^{-2}\int\limits_{\varepsilon_{\rho,\frac{3}{2}}(y)}\sum_{i=1}^{n}\lambda_i(x)\rho^{-\alpha_i}dx. \tag{49}$$

On the other side, arguing the same, as well as in the proof of Lemma 10 we came lead to the inequality

$$\lambda_i(x)<C_{19}\left(\alpha,n\right)\left(R+\rho\right)^{\alpha_i}, \qquad x\in\varepsilon_{\rho,\frac{3}{2}}(y), \quad i=1,\ldots,n. \tag{50}$$

Now, it is sufficient to take into account that

$$\sum_{i=1}^{n}\left(1+\frac{R}{\rho}\right)^{\alpha_i}\le\sum_{i=1}^{n}\left[1+\left(\frac{R}{\rho}\right)^{\alpha_i}\right]\le n\left(1+\sum_{i=1}^{n}\left(\frac{R}{\rho}\right)^{\alpha_i}\right),$$

and from (49), (50) follows the required estimation (48).

***Corollary* 3.** *If conditions* (1), (2) $y\neq0$, *be fulfilled, then at* $x\in\varepsilon_{d|y|_d,1}(y)$, $x\neq y$ *for the fundamental solution* $G(x,y)$ *the estimation*

$$G(x,y)\ge C_{20}\left(\gamma,\alpha,n\right)\frac{\left(|x-y|_\alpha\right)^{2-n-\frac{\langle\alpha\rangle}{2}}}{1+\sum_{i=1}^{n}\left(\dfrac{|y|_\alpha}{|x-y|_\alpha}\right)^{\alpha_i}} \tag{51}$$

is true.

If $y=0$, then estimation (51) is true for all $x\neq0$. Here $d=\dfrac{1}{n2^{\frac{2}{2+\alpha}}}$.

For proving, at first let's show, that if $y\neq0$, then $y\notin\varepsilon_{d|y|_d,2}(0)$. Really, as

$$|y|_\alpha=\sum_{i=1}^{n}|y_i|^{\frac{2}{2+\alpha i}}, \tag{52}$$

then there exists i_0, $1\le i_0\le n$, such that

$$|y_0|^{\frac{2}{2+\alpha i_0}}\ge\frac{|y|_\alpha}{n}.$$

Thus

$$\frac{|y_{i_0}^2|}{\left(|y|_\alpha\right)^{\alpha i_0}}\ge\frac{\left(|y|_\alpha\right)^2}{n^{2+\alpha i}}.$$

There by

$$\sum_{i=1}^{n}\frac{y_i^2}{\left(d\,|y|_\alpha\right)^{\alpha_i}}\ge\frac{y_{i_0}^2}{\left(d\,|y|_\alpha\right)^{\alpha_{i_0}}}\ge\frac{\left(d\,|y|_\alpha\right)^2}{(dn)^{2+\alpha i_0}}=\frac{4\left(d\,|y|_\alpha\right)^2}{\left(2^{\frac{2}{2+\alpha i_0}}dn\right)^{2+\alpha i_0}}.$$

Now, it is sufficient to note that $2^{\frac{2}{2+\alpha i_0}}dn\le2^{\frac{2}{2+\alpha}}dn=1$ and the required assertion is proved. On the other side from (52) it follows that for all i, $1\le i\le n$,

$$|y_i|^{\frac{2}{2+\alpha_i}}\le|y|_\alpha,$$

i.e.,

$$\sum_{i=1}^{n} \frac{y_i^2}{\left(|y|_\alpha\right)^{\alpha_i}} \leq n\left(|y|_\alpha\right)^2.$$

So, we'll show that $\varepsilon_{|y|\alpha,\sqrt{n}}(0)$, if only $y \neq 0$.

Let now, for $y \neq 0$, $x \in \varepsilon_{d|y|_d,1}(y)$ and $x \neq y$. Denote by $|x-y|_\alpha$ the ρ. It is easy to see that there exists i_1, $1 \leq i_1 \leq n$, such that

$$|x_{i_1} - y_{i_1}|^{\frac{2}{2+\alpha i_1}} \geq \frac{\rho}{n}.$$

Hence, it follows that

$$\sum_{i=1}^{n} \frac{(x_i - y_i)^2}{\rho^{\alpha_i}} \geq \frac{(x_{i_1} - y_{i_1})^2}{\rho^{\alpha_1}} \geq \frac{\rho^2}{n^{2+\alpha i_1}} \geq \frac{\rho^2}{n^{2+\alpha}}.$$

Thus $x \notin \varepsilon_{\rho;d_1}(y)$, where $d_1 = \dfrac{1}{n^{1+\frac{\alpha}{2}}}$. Analogously, it is proved that $x \in \varepsilon_{\rho,\sqrt{n}}(y)$. Now, the required estimation (51) at $y \neq 0$ follows from (27) and Corollary 1 from Lemma 10. If $y = 0$, then (51), it immediately follows from (27) and Lemma 7.

Let $F(x,y)$ be a positive function, determined in $E_n \times E_n$, continuous at $x \neq y$, moreover $\lim_{x \to y} F(x,y) = \infty$ (condition (A)).

Further, let $E \subset E_n$ be some compact. Let's call the measure μ on E $[F]$ admissible, if $\sup p\mu \subset E$ and $V_\mu^E(x) = \displaystyle\int_E F(x,y)d\mu(y) \leq 1$, for $x \in \sup p\mu$.

The value $\sup \mu(E) = \operatorname{cap}_{[F]}(E)$, where an exact upper boundary is taken by all $[F]$ admissible measures, is called $[F]$-capacity of the compact E.

Theorem 2. *Let relative to the coefficients of the operator $\mathcal{L}$ conditions* (1), (2) *be fulfilled. Then for removability of the compact $E \subset D$ relative to the first boundary-value problem for the operator $\mathcal{L}$ in the space $\mathcal{M}(D)$ it is sufficient that*

$$\operatorname{cap}_{[F_1]}(E) = 0, \tag{53}$$

where

$$F_1(x,y) = \left[1 + \sum_{i=1}^{n} \left(\frac{|y|\,\alpha}{|x-y|_\alpha}\right)^{\alpha_i}\right]^{-1} \left(|x-y|_\alpha\right)^{2-n-\frac{\langle\alpha\rangle}{2}}.$$

Proof. We'll use the following assertion, which is proved in [10]. Let the function $F(x,y)$ be satisfied condition (A), the compact E has zero $[F]$-capacity, μ zero measure concentrated on E. Then, there exists the point $x^\circ \in \sup p\mu$, such that $V_\mu^E(x^\circ) = \infty$. At this the potential of the measure $\sup p\mu$ can't be bounded on any portion B, i.e., for any open set B at $E' \in \sup p\mu \operatorname{cap} B$, the potential $V_\mu^{E'}(x)$ is not bound B. In particular, if B is n arbitrary neighborhood of the point x° that $V_\mu^{E'}(x^\circ) = \infty$.

Let the condition (53) be fulfilled, μ be an arbitrary measure, concentrated on E, $x^\circ \in \sup p\mu$ is a point, corresponding to the above-stated assertion at $F = F_1$. Let's assume at first, that $x^\circ \neq 0$. Then $|x^\circ|_\alpha = v > 0$. Further, let B be such small neighborhood of the point x°, that if $E' \in \sup p\mu \operatorname{cap} B$, then

$$\sup_{y \in E'} |y|_\alpha \leq (1+\varepsilon)\,r, \qquad \inf_{y \in E'} |y|_\alpha \geq (1+\varepsilon)\,r,$$

where the number $\varepsilon > 0$ will be chosen later. Let's consider the ellipsoids $\varepsilon_{d|y|_d,1}(y)$ at $y \in E'$. Let's choose ε such small, than $x^0 \in \varepsilon_{d|y|_d,1}(y)$ for all $y \in E'$. Then according to Corollary 2 from Lemma 7 we obtain

$$V_\mu^E\left(x^0\right) = \int\limits_E G\left(x^0, y\right) d\mu(y) \geq \int\limits_{E'} G\left(x^0, y\right) d\mu(y) \geq$$

$$\geq C_{20} \int\limits_E F_1\left(x^0, y\right) d\mu(y) = C_{20} V_\mu^E\left(x^0\right) = \infty.$$

Hence, it follows that any zero measure μ, concentrated on E can't be $\mathcal{L}$ admissible. Thus $\operatorname{cap}_{\mathcal{L}}(E) =$ $= 0$ and the required assertion is follows from Theorem 1.

Let now $x^\circ = 0$. Then, using the equality $G(x, y) = G(y, x)$ and Corollary 2 from Lemma 7 we conclude

$$V_\mu^E(0) \int\limits_E G(0, y)\, d\mu(y) = \int\limits_E G(y, 0)\, d\mu(y) \geq C_{20} \int\limits_E F_1(y, 0)\, d\mu(y) =$$

$$= C_{20} \int\limits_E F_1(0, y)\, d\mu(y) = C_{20} V_\mu^E(0) = \infty.$$

Theorem 2 is proved.

Remark. Let conditions of the real theorem be fulfilled, and the compact $E \subset D$ is removable relative to the first boundary-value problem for the operator $\mathcal{L}$ in the space $\mathcal{M}(D)$. Then $\operatorname{mes}(E) = 0$.

At first, let's note for proofing that the discussion are the same, as at conclusion of estimation (51), we can show that at $x \in \varepsilon_{d|y|_d,1}(y)$, $x \neq y$ $(y \neq 0)$ and at $x \neq y$ $(y = 0)$ the estimations

$$G(x, y) \leq C_{21}(\gamma, \alpha, n) \left(|x - y|_d\right)^{2-n-\frac{\langle\alpha\rangle}{2}} \tag{54}$$

are true.

Further, analogously to Theorem 6, it is shown that if the compact E is removable, then according to $\operatorname{cap}_{[-F_2]}(E) = 0$, where $F_2(x, y) = \left(|x - y|_d\right)^{2-n-\frac{\langle\alpha\rangle}{2}}$.

Hence, it follows that if $\operatorname{mes}(E) > 0$, then there exists the point $x^2 \in E$, such that $V^E\left(x^1\right) = \infty$, where

$$V^E(x) = \int\limits_E F_2(x, y) dy.$$

Moreover, if B' is an arbitrary neighborhood of the point $E' = B' \operatorname{cap} E$, then the potential $V^{E'}(x)$ is not bounded on E'. Let's consider the case $x^1 \neq 0$. Choose small neighborhood B' of the point x^1, that at all $x \in E'$, $y \in E'$ the inequality $|x_i - y_i| \leq 1$, $i = 1, \ldots, n$, are fulfilled. For $x \in E'$ we have

$$V^{E'}(x) = \int\limits_{E'} \left(\sum_{i=1}^n |x_i - y_i|^{\frac{2}{2+\alpha i}}\right)^{2-n-\frac{\langle\alpha\rangle}{2}} dy \leq \int\limits_{E'} \left(\sum_{i=1}^n |x_i - y_i|\right)^{2-n-\frac{\langle\alpha\rangle}{2}} dy \leq$$

$$\leq \int\limits_{E'} |x-y|^{2-n-\frac{\langle\alpha\rangle}{2}}\,dy \leq \int\limits_{B''} |z|^{2-n-\frac{\langle\alpha\rangle}{2}}\,dy,$$

where B'' is a ball of the radius $\sqrt{n}$ with the center origin of the coordinate. Now, it is sufficient to note that according to condition (2) $\dfrac{\langle\alpha\rangle}{2} \leq \dfrac{n}{n-1} \leq \dfrac{3}{2}$ and the assertion the corollary is proved.

1. *Carleson L.* Removable singularities of continuous harmonic functions in R^n // Math. scand. – 1963. – **12**. – P. 15 – 18.
2. *Moiseev E. I.* On existence and nonexistence of boundary sets of Neumann problem // Different. Equat. – 1973. – **9**, № 5. – P. 901 – 911.
3. *Landis E. M.* To the question on uniqueness of solution of the first boundary-value problem for elliptic and parabolic equations of second order // Uspechi mat. nauk. – 1978. – **33**, № 3. – 151 p.
4. *Gadjiev T. S., Mamedova V. A.* On removable sets of solutions of the second order elliptic and parabolic equations in nondivergent form // Ukr. Math. J. – 2009. – **61**, № 11. – P. 1485 – 1496.
5. *Harvey R., Polking J.* Removable singularities of solutions of linear partial differential equations // Acta Math. – 1970. – **125**. – P. 39 – 56.
6. *Kilpelainen T., Zhong X.,* Removable sets for continuous solutions of quasilinear elliptic equations // Proc. Amer. Math. Soc. – 2002. – **130**, № 6. – P. 1681 – 1688.
7. *Chanillo S., Wreeden R.* Harnack's inequality and mean-value inequalities // Commun. Part Different. Equat. – 1986. – № 11. – P. 1111 – 1134.
8. *Littman W., Stampacchia G., Weinberger H.* Regular points for elliptic equations with discontinuous coefficients // Ann. Scuola norm. super. Pisa. – 1963. – **17**. – P. 45 – 79.
9. *Schwartz L.* Theorie des Distributions. – Paris: Hermann, 1950, 1951. – Vols 1, 2.
10. *Landkof N. S.* Foundations of modern potential theory. – Moscow: Nauka, 1966.

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