UDC 517.9

A. Kharazishvili (A. Razmadze Math. Inst., Tbilisi, Georgia)

ON BIJECTIVE CONTINUOUS IMAGES OF ABSOLUTE NULL SETS

ПРО ВЗАЄМНО ОДНОЗНАЧНІ НЕПЕРЕРВНІ ВІДОБРАЖЕННЯ МНОЖИН АБСОЛЮТНОЇ МІРИ НУЛЬ

The images of absolute null sets (spaces) under bijective continuous mappings are studied. It is shown that, in general, such images do not possess regularity properties from the viewpoint of topological measure theory.

Вивчаються зображення множин (просторів) абсолютної міри нуль при взаємно однозначних відображеннях. Доведено, що (в загальному випадку) ці зображення не мають властивостей регулярності з точки зору топологічної теорії міри.

There are various concepts of so-called small sets (spaces). Classical examples of such sets are Luzin subsets of the real line $\mathbf{R}$ and Sierpiński subsets of the same line (see, e. g., [1 – 5]). Here we would like to discuss one concept of such spaces which is motivated by topological measure theory.

All topological spaces E considered below are assumed to have the following property: for any point $x \in E$, the singleton $\{x\}$ is a Borel set in E. In particular, any Hausdorff topological space has this property.

A measure μ defined on some σ-algebra $\mathcal{S}$ of subsets of E is called diffused (or continuous) if $\{x\} \in \mathcal{S}$ and $\mu(\{x\}) = 0$ for each point $x \in E$.

According to a well-known definition, E is said to be an absolute null space (or universal measure zero space) if every Borel σ-finite diffused measure on E is identically equal to zero.

***Example* 1.** Every Luzin subset of $\mathbf{R}$ (and, under Martin's Axiom, every generalized Luzin subset of $\mathbf{R}$) is an absolute null space. Also, without assuming additional set-theoretical hypotheses, there exist uncountable absolute null subspaces of $\mathbf{R}$ (see, for instance, [2, 3, 6, 8]). Moreover, any nonempty perfect subset of $\mathbf{R}$ contains an uncountable absolute null space.

***Example* 2.** Let E be an infinite set equipped with the discrete topology and let E' denote Alexandrov's one-point compactification of E. Then these two assertions are equivalent:

(a) E' is an absolute null space;

(b) the cardinal number $\mathrm{card}(E')$ is nonmeasurable in the Ulam sense.

We thus see that there exist compact absolute null spaces whose cardinalities are sufficiently large. Some other examples of nondiscrete Hausdorff absolute null spaces with sufficiently large cardinalities can be found in [6] and [7].

It readily follows from the definition of absolute null spaces that:

(i) the topological product of finitely many absolute null spaces is also an absolute null space;

(ii) if E' is an absolute null space and $h\colon E \to E'$ is a Borel mapping such that $h^{-1}(y)$ is at most countable for each point $y \in E'$, then E is an absolute null space (in particular, any subspace of an absolute null space is also an absolute null space);

(iii) if $\{E_j : j \in J\}$ is a family of absolute null spaces such that the cardinal number $\mathrm{card}(J)$ is nonmeasurable in the Ulam sense, then the topological sum of this family is an absolute null space;

(iv) if $\{E_j : j \in J\}$ is a countable family of absolute null subspaces of E, then $\cup\{E_j : j \in J\}$ is also an absolute null subspace of E;

(v) if X is a Hausdorff absolute null space and Y is a Radon space, then the topological product $X \times Y$ is a Radon space;

(vi) every absolute null subspace of $\mathbf{R}$ is zero-dimensional.

Notice, in connection with (i), that the topological product of countably many absolute null spaces is not, in general, an absolute null space.

One can readily deduce from (ii) that if E is an absolute null space and $f\colon E \to E'$ is a Borel isomorphism, then E' is an absolute null space. In particular, the class of absolute null spaces is closed under the operation of taking homeomorphic images.

***Example* 3.** If X is a Luzin subset of $\mathbf{R}$ and $Y \subset \mathbf{R}$ is a homeomorphic image of X, then one cannot assert that Y is a Luzin set. Moreover, such a Y can be a nowhere dense subset of $\mathbf{R}$. Analogously, if X' is a Sierpiński subset of $\mathbf{R}$ and $Y' \subset \mathbf{R}$ is a homeomorphic image of X', then one cannot assert that Y' is a Sierpiński set. Moreover, such a Y' can be a λ-measure zero subset of $\mathbf{R}$, where λ stands for the standard Lebesgue measure on $\mathbf{R}$. On the other hand, it is easy to see that if $f:\mathbf{R} \to \mathbf{R}$ is a bijection such that both f and f^{-1} preserve the σ-ideal of all first category sets in $\mathbf{R}$, then f transforms the class of all Luzin subsets of $\mathbf{R}$ onto it-self. Similarly to this fact, if $g\colon \mathbf{R} \to \mathbf{R}$ is a bijection such that both g and g^{-1} preserve the σ-ideal of all λ-measure zero sets, then g transforms the class of all Sierpiński subsets of $\mathbf{R}$ onto it-self.

***Example* 4.** If X is a Luzin set and $f\colon X \to \mathbf{R}$ is a mapping having the Baire property, then $f(X)$ is an absolute null space (hence $f(X)$ is totally imperfect as well). Analogously, if Y is a Sierpiński set and a mapping $g\colon Y \to \mathbf{R}$ is measurable with respect to the measure on Y induced by λ, then $g(Y)$ does not contain any uncountable absolute null space (hence $g(Y)$ is totally imperfect as well).

Theorem 1. *Let E be a compact absolute null space, E' be a Hausdorff space, and let $\phi\colon E \to E'$ be a mapping such that the ϕ-pre-image of any open subset of E' is of type G_δ in E (or, equivalently, the ϕ-pre-image of any closed subset of E' is of type F_σ in E).*

Then either $\phi(E)$ is an absolute null space or $\phi(E)$ is not a Radon space.

Proof. We may assume, without loss of generality that ϕ is a surjection, i. e., $E' = \phi(E)$. If E' is an absolute null space, then there is nothing to prove. Suppose now that E' is not an absolute null space, so there exists a probability diffused Borel measure μ on E'. If this μ is not a Radon measure, then E' is not a Radon space, and we are done. So it remains to consider the case when μ is a Radon measure. In this case, let us put

$$\mathcal{S} = \{\phi^{-1}(Y)\colon Y \in \mathcal{B}(E')\}.$$

The introduced class $\mathcal{S}$ of subsets of E is a σ-subalgebra of the Borel σ-algebra $\mathcal{B}(E)$, and we can define a probability measure ν on $\mathcal{S}$ by putting:

$$\nu(\phi^{-1}(Y)) = \mu(Y) \qquad (Y \in \mathcal{B}(E')).$$

Let us check that this ν is a Radon probability measure on $\mathcal{S}$. For this purpose, take any set $Y \in \mathcal{B}(E')$. Since μ is a Radon measure, we may write

$$\mu(Y) = \sup\{\mu(K_j)\colon j \in J\},$$

where J is a countable set of indices and, for each $j \in J$, the set $K_j \subset Y$ is compact. It directly follows from the above equality that

$$\nu\big(\phi^{-1}(Y)\big) = \sup\big\{\nu(\phi^{-1}(K_j)) : j \in J\big\},$$

where all sets $\phi^{-1}(K_j)$ $(j \in J)$ are of type F_σ in E and all of them belong to $\mathcal{S}$ and are contained in $\phi^{-1}(Y)$. Taking into account the compactness of E and the assumption on ϕ, all sets $\phi^{-1}(K_j)$ $(j \in J)$ are σ-compact in E. This circumstance directly implies that ν is a Radon measure on $\mathcal{S}$. Now, according to the well-known Henry's theorem, there exists a Radon probability measure ν' extending ν and defined on the whole Borel σ-algebra $\mathcal{B}(E)$. Clearly, the measure ν' is diffused. So we come to a contradiction with the fact that E is an absolute null space.

The obtained contradiction finishes the proof.

***Example* 5.** Let ω_1 denote the least uncountable ordinal number, $E = X \cup \{x\}$ denote Alexandrov's one-point compactification of a discrete topological space X of cardinality ω_1, where $x \notin X$, and let the closed interval $E' = [0, \omega_1]$ of ordinal numbers be equipped with its order topology. As is well known, E' carries a Borel probability diffused measure ν (the so-called Dieudonné measure) and this ν is not Radon. Also, by virtue of Ulam's classical theorem (see, e. g., [4, 5, 8]), E is an absolute null space. Consider the mapping $\phi\colon E \to E'$ such that $\phi(x) = \omega_1$ and the restriction $\phi|X$ is a bijection between X and $[0, \omega_1[$. Then ϕ satisfies the assumption of Theorem 1 and the space $E' = \phi(E)$ is not Radon.

***Remark* 1.** Theorem 1 and Example 5 show that the images of absolute null spaces may lack natural measure-theoretical regularity properties even under quite good mappings.

In what follows, we are going to demonstrate that, similarly to Remark 1, among the bijective continuous images of absolute null subspaces of $\mathbf{R}$, one may meet those ones which possess very bad properties from the point of view of topological measure theory. For this purpose, we need the notion of absolutely nonmeasurable sets.

Let E be a base (ground) set and let $\mathcal{M}$ be a class of measures on E (in general, the domains of measures from $\mathcal{M}$ are diverse σ-algebras of subsets of E).

We say that a function $f : E \to \mathbf{R}$ is absolutely nonmeasurable with respect to $\mathcal{M}$ if there exists no measure $\mu \in \mathcal{M}$ such that f is μ-measurable. Accordingly, we say that a set $Z \subset E$ is absolutely nonmeasurable with respect to $\mathcal{M}$ if the characteristic function

$$\chi_Z\colon E \to \{0, 1\}$$

is absolutely nonmeasurable with respect to $\mathcal{M}$.

***Example* 6.** Any Bernstein subset of $\mathbf{R}$ is absolutely nonmeasurable with respect to the class of completions of all nonzero σ-finite diffused Borel measures on $\mathbf{R}$. Moreover, it can be proved that there exists a Bernstein set in $\mathbf{R}$ which is absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation quasi-invariant measures on $\mathbf{R}$. Many interesting and extraordinary properties of Bernstein sets are considered in [1 – 5] and [8].

To proceed, we need four auxiliary propositions. The first of them is due to Erdös, Kunen, and Mauldin [9].

Lemma 1. *Under Martin's Axiom, there exist two generalized Luzin sets L_1 and L_2 satisfying the following two conditions*:

(1) *both L_1 and L_2 are vector spaces over the field $\mathbf{Q}$ of all rational numbers*;

(2) *the real line $\mathbf{R}$ considered as a vector space over $\mathbf{Q}$ is a direct sum of L_1 and L_2.*

The proof of Lemma 1 may be found in [9] (see also [10] for a more general result).

Lemma 2. *Under Martin's Axiom, there exists a Hamel basis of* $\mathbf{R}$ *which simultaneously is a generalized Luzin set.*

Proof. We follow the idea of Sierpiński [11]. Let L_1 and L_2 be as in Lemma 1. Then the set $L = L_1 \cup L_2$ is a generalized Luzin set in $\mathbf{R}$. Consider a maximal (with respect to the inclusion relation) linearly independent (over $\mathbf{Q}$) subset H of L. The existence of H immediately follows from the widely known Kuratowski – Zorn lemma. We assert that H is a Hamel basis of $\mathbf{R}$. Suppose otherwise, i.e., there is an element $x \in \mathbf{R}$ such that $x \notin \mathrm{span}_{\mathbf{Q}}(H)$. Since $\mathbf{R}$ is a direct sum of L_1 and L_2. we may write

$$x = l_1 + l_2, \quad l_1 \in L_1, \quad l_2 \in L_2.$$

Since $x \notin \mathrm{span}_{\mathbf{Q}}(H)$, we have either $l_1 \notin \mathrm{span}_{\mathbf{Q}}(H)$ or $l_2 \notin \mathrm{span}_{\mathbf{Q}}(H)$. We may assume, without loss of generality, that $l_1 \notin \mathrm{span}_{\mathbf{Q}}(H)$. Then the set $\{l_1\} \cup H$ is linearly independent over $\mathbf{Q}$, is contained in L and properly contains H. But this contradicts the maximality of H. The obtained contradiction completes the proof.

As usual, we denote by $\mathbf{c}$ the cardinality of the continuum.

Lemma 3. *Assume Martin's Axiom and let* E *be a separable metric space equipped with the completion of some nonzero* σ*-finite diffused Borel measure on* E. *Then there exists a set* $A \subset E$ *such that* $\mathrm{card}(A) = \mathbf{c}$ *and no subset of* A *having the same cardinality* $\mathbf{c}$ *is an absolute null space.*

Proof. The argument is very similar to the classical Sierpiński construction producing Sierpiński subsets of $\mathbf{R}$ (cf. [2 – 5] or [8]), so we omit it here.

Lemma 4. *Assume Martin's Axiom and let* E *be an arbitrary metric space whose cardinality does not exceed* $\mathbf{c}$. *Then there exists a subset* B *of* E *which is absolutely nonmeasurable with respect to the class* $\mathcal{M}$ *of completions of all nonzero* σ*-finite diffused Borel measures on* E.

Proof. If E is an absolute null space, then there is nothing to prove. So suppose that $\mathcal{M} \neq \varnothing$. In this case, the cardinality of E is equal to $\mathbf{c}$ and all subsets of E whose cardinalities are strictly less than $\mathbf{c}$ are absolute null subspaces of E. Now, we may apply Bernstein's classical transfinite construction (cf. [1 – 5] or [8]) to the family of all closed separable subsets of E having cardinality $\mathbf{c}$. By virtue of this construction, there exists a set $B \subset E$ such that, for any closed separable subset F of E with $\mathrm{card}(F) = \mathbf{c}$, the equalities

$$\mathrm{card}(F \cap B) = \mathrm{card}(F \cap (E \setminus B)) = \mathbf{c}$$

are fulfilled. Then it is not difficult to show that the set B is absolutely nonmeasurable with respect to the class $\mathcal{M}$. Lemma 4 has thus been proved.

***Remark* 2.** Let us underline that in the above proof the completeness of E is not needed. In the literature the existence of Bernstein sets is usually established for uncountable Polish spaces (cf. [1 – 5]).

Theorem 2. *Under Martin's Axiom, there exist an absolute null subspace* Z *of* $\mathbf{R}$ *with* $\mathrm{card}(Z) = \mathbf{c}$ *and a continuous injection* $\phi \colon Z \to \mathbf{R}$ *such that no subset of* $\phi(Z)$ *having cardinality* $\mathbf{c}$ *is absolute null.*

Proof. Let L_1 and L_2 be as in Lemma 1, i. e., both L_1 and L_2 are generalized Luzin sets, are vector spaces over the field $\mathbf{Q}$, and the real line $\mathbf{R}$ also considered as a vector space over $\mathbf{Q}$ is a direct sum of L_1 and L_2. Let us introduce the product set

$$L = L_1 \times L_2 \subset \mathbf{R}^2$$

and define a mapping $\phi\colon L \to \mathbf{R}$ by the formula

$$\phi(z) = \mathrm{pr}_1(z) + \mathrm{pr}_2(z) \qquad (z \in L_1 \times L_2).$$

According to (i), the product set L is an absolute null subspace of $\mathbf{R}^2$. Moreover, L being the product of two zero-dimensional spaces is a zero-dimensional subspace of $\mathbf{R}^2$ (see (vi)). Since $\mathbf{R}$ is a direct sum of L_1 and L_2, the additive mapping ϕ is a continuous bijection of L onto $\mathbf{R}$. Keeping in mind the circumstance that $\mathbf{R}$ is equipped with the nonzero σ-finite diffused Lebesgue measure λ, we may apply Lemma 3 to $E = \mathbf{R}$. Consequently, there exists a set $A \subset \mathbf{R}$ with $\mathrm{card}(A) = \mathbf{c}$ such that no subset of A having the same cardinality $\mathbf{c}$ is an absolute null space (actually, A can be a generalized Sierpiński subset of $\mathbf{R}$). Now, put $Z = \phi^{-1}(A)$ and observe that $\mathrm{card}(Z) = \mathbf{c}$ and Z, being a subset of L, is a zero-dimensional subspace of $\mathbf{R}^2$. Finally, identifying Z with its topological copy in $\mathbf{R}$, we obtain the required result.

In a similar manner, we are able to prove the following statement.

Theorem 3. *Under Martin's Axiom, there exist an absolute null subset Y of $\mathbf{R}$ with $\mathrm{card}(Y) = \mathbf{c}$, a continuous injection $\phi\colon Y \to \mathbf{R}$, such that the set $\phi(Y)$ is absolutely nonmeasurable with respect to the class of completions of all nonzero σ-finite diffused Borel measures on $\mathbf{R}$.*

Proof. The argument is analogous to the previous one. We preserve the notation of the proof of Theorem 2. Let L and ϕ be as earlier. According to Lemma 4, there exists a set $B \subset \mathbf{R}$ absolutely nonmeasurable with respect to the class of completions of all nonzero σ-finite diffused Borel measures on $\mathbf{R}$ (actually, B is a Bernstein subset of $\mathbf{R}$). Let us put $Y = \phi^{-1}(B)$. Then $\mathrm{card}(Y) = \mathbf{c}$ and $Y \subset L$ is a zero-dimensional absolute null subspace of $\mathbf{R}^2$. Identifying Y with its topological copy in $\mathbf{R}$, we obtain the required result.

***Remark* 3.** As is well known, the existence of an absolute null subspace of $\mathbf{R}$ of cardinality $\mathbf{c}$ cannot be proved within the framework of **ZFC** set theory. Moreover, there is a model of **ZFC** in which $\omega_1 < \mathbf{c}$ and the cardinalities of all absolute null subspaces of $\mathbf{R}$ do not exceed ω_1 (see, e.ġ., [3] and references therein). Therefore, additional set-theoretical assumptions are necessary in the formulations of Theorems 2 and 3.

1. *Kuczma M.* An introduction to the theory of functional equations and inequalities: cauchy's equation and jensen's inequality. – Katowice: PWN, 1985.
2. *Kuratowski K.* Topology. – London; New York: Acad. Press, 1966. – Vol. 1.
3. *Miller A. W.* Special subsets of the real line // Handbook of Set-Theoretic Topology / Eds K. Kunen and J. E. Vaughan. – Amsterdam: North-Holland Publ. Co., 1984.
4. *Morgan II J. C.* Point set theory. – New York: Marcel Dekker, Inc., 1990.
5. *Oxtoby J. C.* Measure and category. – New York: Springer-Verlag, 1971.
6. *Pfeffer W. F., Prikry K.* Small spaces // Proc. London Math. Soc. – 1989. – **58(3)**, № 3. – P. 417 – 438.
7. *Kharazishvili A. B.* Some properties of isodyne topological spaces // Bull. Acad. Sci. GSSR. – 1987. – **127**, № 2. – P. 261 – 264 (in Russian).
8. *Kharazishvili A. B.* Nonmeasurable sets and functions. – Amsterdam: Elsevier, 2004.
9. *Erdös P., Kunen K., Mauldin R. D.* Some additive properties of sets of real numbers // Fund. Math. – 1981. – **113**, № 3. – P. 187 – 199.
10. *Kharazishvili A. B.* Sums of absolutely nonmeasurable functions // Georg. Math. J. – 2013. – **20**, № 2. – P. 271 – 282.
11. *Sierpiński W.* Sur la question de la mesurabilité de la base de M.Hamel // Fund. Math. – 1920. – **1**. – P. 105 – 111.

Received 12.06.14