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# RINGS WHOSE NONSINGULAR MODULES HAVE PROJECTIVE COVERS

# КІЛЬЦЯ, ДЛЯ ЯКИХ НЕСИНГУЛЯРНІ МОДУЛІ МАЮТЬ ПРОЕКТИВНІ ПОКРИТТЯ

We determine rings $R$ with the property that all (finitely generated) nonsingular right $R$-modules have projective covers. These are just the rings with $t$-supplemented (finitely generated) free right modules. Hence, they are called *right (finitely)* $\Sigma$-$t$-*supplemented*. It is also shown that a ring $R$ for which every cyclic nonsingular right $R$-module has a projective cover is exactly a right $t$-supplemented ring. It is proved that, for a continuous ring $R$, the property of right $\Sigma$-$t$-supplementedness is equivalent to the semisimplicity of $R/Z_2(R_R)$, while the property of being right finitely $\Sigma$-$t$-supplemented is equivalent to the right self-injectivity of $R/Z_2(R_R)$. Moreover, for a von Neumann regular ring $R$, the properties of being right $\Sigma$-$t$-supplemented, right finitely $\Sigma$-$t$-supplemented, and right $t$-supplemented are equivalent to the semisimplicity, right self-injectivity, and right continuity of $R$, respectively.

Визначено кільця $R$ з тією властивістю, що всі (скінченнопороджені) несингулярні праві $R$-модулі мають проективні покриття. Це є саме кільця з $t$-доповненими (скінченнопородженими) вільними правими модулями. Таким чином, вони називаються *правими* (*скінченно*) $\Sigma$-$t$-*доповненими*. Також показано, що кільце $R$, для якого кожний циклічний несингулярний правий $R$-модуль має проективне покриття, є в точності правим $t$-доповненим кільцем. Доведено, що для скінченного кільця $R$ властивість правої $\Sigma$-$t$-доповненості еквівалентна напівпростоті $R/Z_2(R_R)$, а властивість правої скінченної $\Sigma$-$t$-доповненості — правій самоін'єктивності $R/Z_2(R_R)$. Крім того, для регулярного кільця фонНоймана $R$ властивості правої $\Sigma$-$t$-доповненості, правої скінченної $\Sigma$-$t$-доповненості та правої $t$-доповненості еквівалентні відповідно напівпростоті, правій самоін'єктивності та правій неперервності $R$.

**1. Introduction.** Let $R$ be a ring and $\mathcal{C}$ be a class of right $R$-modules. For some special classes $\mathcal{C}$, the property of having a projective cover for each element of $\mathcal{C}$ characterizes $R$. In [5], Bass studied the rings $R$ for which every element of $\mathcal{C}$ has a projective cover, when $\mathcal{C}$ is the class of all right $R$-modules (resp., cyclic right $R$-modules). He called such rings right perfect rings (resp., semiperfect rings). An excellent reference for a thorough study of these rings and their applications is [14]. When $\mathcal{C}$ is the class of semisimple right $R$-modules, each element of $\mathcal{C}$ has a projective cover, if and only if, $R$ is right perfect; see [21] (43.9) and [17] (Theorem B.38). If $\mathcal{C}$ is either the class of finitely generated $R$-modules or the class of simple $R$-modules, then each element of $\mathcal{C}$ has a projective cover, if and only if, $R$ is semiperfect [21] (42.6); and by [7] (Proposition 2.6), these are equivalent to $R$ being lifting. In [4], Azumaya called a ring $R$ F-semiperfect if $R/\operatorname{Rad}(R)$ is von Neumann regular and idempotents can be lifted modulo $\operatorname{Rad}(R)$. F-semiperfect rings are also known as semiregular rings. If $\mathcal{C}$ is the class of all factor modules $R/I$ where $I$ is a principal (finitely generated) right ideal of $R$, then each element of $\mathcal{C}$ has a projective cover, if and only if, $R$ is semiregular; see [4] (Proposition 1.7) and [17] (Theorem B.44). Moreover, when $\mathcal{C}$ is the class of all singular right $R$-modules, Guo in [12] showed that every element of $\mathcal{C}$ has a projective cover, if and only if, $R$ is right perfect. So a natural question is: When $\mathcal{C}$ is the class of all nonsingular right $R$-modules,

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which rings are determined by having the property that each element of $\mathcal{C}$ has a projective cover? We are more interested in characterizing such rings in a way similar to the characterization in [2] of rings whose nonsingular modules are projective. We approach this by restricting the $\oplus$-supplemented property to the $t$-closed submodules of projective modules, which we define in the following.

Throughout the paper, rings will have a nonzero identity element and modules will be unitary right modules. Recall that a submodule $K$ of a module $M$ is called a supplement or 'addition complement' of a submodule $A$ if $K$ is minimal with respect to the property that $A + K = M$. Indeed, $K$ is a supplement of $A$, if and only if, $A + K = M$ and $A \cap K \ll K$ (the notation $\ll$ denotes a small submodule). A module $M$ is called supplemented if any submodule $A$ of $M$ has a supplement in $M$, and is called $\oplus$-supplemented if any submodule $A$ of $M$ has a supplement in $M$ which is a direct summand. For a projective module, the properties of supplemented and $\oplus$-supplemented are equivalent. A $t$-closed submodule $C$ of a module $M$ (denoted by $C \leq_{tc} M$) is introduced in [2] as a closed submodule of $M$ which contains $Z_2(M)$. A module $M$ is called $t$-extending if every $t$-closed submodule of $M$ is a direct summand, and a ring $R$ is called right $\Sigma$-$t$-extending if all free right $R$-modules are $t$-extending.

We say that a projective module $P$ is $t$-supplemented if every $t$-closed submodule $C$ of $P$ has a supplement in $P$ which is a direct summand. In Section 2 we deal with $t$-supplemented projective modules. Projective modules which are either $t$-extending or supplemented are $t$-supplemented. So right extending rings and right lifting rings are right $t$-supplemented. It will be shown that projective modules admit many characterizations for being $t$-supplemented (Theorem 2.1). The properties of $t$-supplemented and $t$-extending coincide for projective modules with zero radical (Proposition 2.2), and for projective modules over right continuous rings (Corollary 2.5).

In Section 3 we will prove that rings for which nonsingular modules have projective covers are precisely rings whose all free modules are $t$-supplemented, called right $\Sigma$-$t$-supplemented rings (Theorem 3.1). Following [3], a ring $R$ is called right $t$-semisimple if $R/Z_2(R_R)$ is semisimple. In fact, $R$ is right $t$-semisimple, if and only if, every nonsingular $R$-module is injective, if and only if, every nonsingular $R$-module is semisimple. For rings we have

$$\text{right } t\text{-semisimple} \Rightarrow \text{right } \Sigma\text{-}t\text{-extending} \Rightarrow \text{right } \Sigma\text{-}t\text{-supplemented}$$

but none of these implications is reversible. The above properties coincide for a ring $R$ such that $\mathrm{Rad}(R) \leq Z_2(R_R)$ (Proposition 3.3). In particular, for right continuous rings and rings with zero radical, the properties of right $\Sigma$-$t$-supplemented, right $\Sigma$-$t$-extending, and right $t$-semisimple are equivalent (Corollary 3.4). A right self-injective right $\Sigma$-$t$-supplemented ring $R$ such that $Z_2(R_R)$ is either Noetherian or Artinian is exactly a quasi-Frobenius ring (Corollary 3.5). In the sequel, we will see that rings whose finitely generated nonsingular modules have projective covers are exactly right finitely $\Sigma$-$t$-supplemented rings (that is, all finitely generated free modules are $t$-supplemented). Moreover, every nonsingular cyclic $R$-module has a projective cover, if and only if, $R$ is right $t$-supplemented. A right continuous ring $R$ is right finitely $\Sigma$-$t$-supplemented if and only if $R/Z_2(R_R)$ is a right self-injective ring (Theorem 3.2). For a von Neumann regular ring $R$ we obtain that: $R$ is right $\Sigma$-$t$-supplemented if and only if $R$ is semisimple (Corollary 3.6); $R$ is right finitely $\Sigma$-$t$-supplemented if and only if $R$ is right self-injective (Corollary 3.7); $R$ is right $t$-supplemented if and only if $R$ is right continuous (Proposition 3.5). Finally, it is shown that the classes of right

$\Sigma$-$t$-supplemented rings, right finitely $\Sigma$-$t$-supplemented rings and right $t$-supplemented rings are different (Example 3.3).

**2. Projective modules with $t$-supplemented property.** By considering the $\oplus$-supplemented property to the $t$-closed submodules of a projective module we define the following notion.

**Definition 2.1.** *We say that a projective module $P$ is* $t$-supplemented *if any $t$-closed submodule $C$ of $P$ has a supplement in $P$ which is a direct summand*, *i.e.*, *there exists a direct summand $K$ of $P$ such that $P = C + K$ and $C \cap K \ll K$. A ring $R$ is called right $t$-supplemented if the module $R_R$ is $t$-supplemented.*

Projective modules which are either $\oplus$-supplemented or $t$-extending are $t$-supplemented. Hence semiperfect rings, right lifting rings, right extending rings, and right $Z_2$-torsion rings $\big($that is, $Z_2(R_R) = R\big)$ are right $t$-supplemented.

The ring of integers $\mathbb{Z}$ is extending, and so it is $t$-supplemented, yet it is not $\oplus$-supplemented. Hence the properties of $\oplus$-supplemented and $t$-supplemented are different for a projective module. Moreover, the property of $t$-supplemented dose not coincide with the property of $t$-extending, as the next result shows.

**Proposition 2.1.** *Let $R$ be a right perfect right nonsingular ring which is not right Artinian. Then there exists a projective $t$-supplemented $R$-module $P$ which is not $t$-extending.*

***Proof.*** Since $R$ is right perfect, every projective $R$-module is supplemented. Thus by [13] (Lemma 1.2), every projective $R$-module is $\oplus$-supplemented, and so it is $t$-supplemented. However, not every projective $R$-module can be extending, for otherwise, $R$ would be right $\Sigma$-$t$-extending by [2] (Theorem 3.12(6)), hence right Artinian by [10] $\big($12.21((a) $\Leftrightarrow$ (b))$\big)$.

In the following we give examples of rings which satisfy the conditions of Proposition 2.1.

***Example*** **2.1.** Let $D$ be a division ring and $\Lambda$ be an infinite set. Consider the upper triangular matrix ring $R = \begin{pmatrix} D & \bigoplus_\Lambda D \\ 0 & D \end{pmatrix}$. Clearly $\operatorname{Rad}(R) = \begin{pmatrix} 0 & \bigoplus_\Lambda D \\ 0 & 0 \end{pmatrix}$. So $R/\operatorname{Rad}(R)$ is semisimple and $\operatorname{Rad}(R)$ is nilpotent. Hence $R$ is right perfect. Moreover, it is easy to see that $R$ is right nonsingular but not right Artinian.

Similarly the ring $R = \begin{pmatrix} D & \prod_\Lambda D \\ 0 & D \end{pmatrix}$ satisfies the conditions of Proposition 2.1.

The next result gives several equivalent conditions for a $t$-supplemented projective module.

**Theorem 2.1.** *Let $P$ be a projective module. The following statements are equivalent*:

(1) *$P$ is $t$-supplemented.*

(2) *$P/C$ has a projective cover for every $t$-closed submodule $C$ of $P$.*

(3) *Every $t$-closed submodule $C$ of $P$ has a supplement which is projective.*

(4) *Every $t$-closed submodule $C$ of $P$ has a supplement which has a projective cover.*

(5) *Every $t$-closed submodule $C$ of $P$ has a supplement which has also a supplement.*

(6) *For every $t$-closed submodule $C$ of $P$, there is a decomposition $P = A \oplus K$ such that $A \leq C$ and $C \cap K \ll K$.*

***Proof.*** (1) $\Rightarrow$ (6). Let $C$ be a $t$-closed submodule of $P$. By hypothesis there exists a direct summand $K$ of $P$ such that $P = C + K$ and $C \cap K \ll K$. Assume that $\pi\colon P \to P/C$ is the natural epimorphism and $f\colon K \to P/C$ is the small epimorphism with $\ker(f) = C \cap K$. Since $P$ is projective we conclude that there exists a homomorphism $g\colon P \to K$ such that $fg = \pi$. Hence $fg(P) = f(K)$, and so $g(P) + (C \cap K) = K$. Thus $g(P) = K$, and $g$ is an epimorphism. Since $K$ is projective we conclude that there exists a homomorphism $h\colon K \to P$ such that $gh = 1_K$. Thus $P = \ker(g) \oplus h(K)$ and clearly $\ker(g) \leq C$. If we show that $C \cap h(K) \ll h(K)$, then $P = \ker(g) \oplus h(K)$ is the desired decomposition of $P$. Let $C\cap h(K) + X = h(K)$. So $g(C\cap h(K))+g(X) = K$. However, $g(C) \leq C$ hence $(C \cap K) + g(X) = K$, and so $g(X) = K$ since $C \cap K \ll K$. Thus for each $k \in K$ there exists $x \in X$ such that $g(x) = k = gh(k)$. This implies that $x - h(k) \in \ker(g) \cap h(K) = 0$ and so $X = h(K)$.

The implication (6) ⇒ (1) is clear, and the equivalences of (1) ⇔ (2) ⇔ (3) ⇔ (4) ⇔ (5) follow from [4] (Proposition 1.4) and [22] (Proposition 2.1).

***Corollary* 2.1.** *The following statements are equivalent for a projective module* $P$:

(1) $P$ *is* $t$*-supplemented.*

(2) *Every* $t$*-closed submodule* $C$ *of* $P$ *has a supplement* $K$ *such that* $C \cap K$ *is a direct summand of* $C$.

(3) *For every* $t$*-closed submodule* $C$ *of* $P$, *there exist a direct summand* $A$ *of* $P$ *and a small submodule* $B$ *of* $P$ *such that* $C = A \oplus B$.

(4) *For every* $t$*-closed submodule* $C$ *of* $P$, *there exists a direct summand* $A$ *of* $P$ *such that* $A \leq C$ *and* $C/A \ll P/A$.

(5) *For every* $t$*-closed submodule* $C$ *of* $P$, *there exists an idempotent* $e \in \operatorname{End}(P)$ *such that* $eP \leq C$ *and* $(1-e)C \ll (1-e)P$.

***Proof.*** This follows from Theorem 2.1(6) and [9] (22.1).

***Corollary* 2.2.** *If* $P$ *is a projective* $t$*-supplemented module*, *then so is every direct summand of* $P$.

***Proof.*** Let $P = P_1 \oplus P_2$ and $C_1 \leq_{tc} P_1$. Clearly $P/(C_1 \oplus P_2) \cong P_1/C_1$. Thus by [2] (Proposition 2.6(6)), $C_1 \oplus P_2 \leq_{tc} P$ and so by Theorem 2.1(2), $P/(C_1 \oplus P_2)$ hence $P_1/C_1$ has a projective cover. Therefore $P_1$ is $t$-supplemented by Theorem 2.1(2).

Let $U$ and $N$ be submodules of a module $M$. It is said that $U$ respects $N$ if there exists a decomposition $M = A \oplus K$ such that $A \leq N$ and $N \cap K \leq U$. In [19] it is shown that $R$ is a semiperfect ring, if and only if, $\operatorname{Rad}(R)$ respects every right ideal of $R$. Moreover, by [19] (Theorem 28) and [17] (Lemma B.40), $R$ is a semiregular ring, if and only if, $\operatorname{Rad}(R)$ respects every finitely generated (principal) right ideal of $R$. The next result shows that a ring $R$ is right $t$-supplemented, if and only if, $\operatorname{Rad}(R)$ respects every $t$-closed right ideal of $R$.

***Corollary* 2.3.** *Let* $P$ *be a projective module such that* $\operatorname{Rad}(P) \ll P$. *The following statements are equivalent*:

(1) $P$ *is* $t$*-supplemented.*

(2) $\operatorname{Rad}(P)$ *respects every* $t$*-closed submodule of* $P$.

***Proof.*** The implication (1) ⇒ (2) is clear by Theorem 2.1(6), and the implication (2) ⇒ (1) follows from Corollary 2.1(3) and [20] (Lemma 3.1).

The next result shows that the properties of $t$-extending and $t$-supplemented coincide for a projective module with zero radical.

**Proposition 2.2.** *If* $P$ *is a* $t$*-supplemented projective module*, *then* $P/\operatorname{Rad}(P)$ *is* $t$*-extending.*

***Proof.*** Let $C/\operatorname{Rad}(P) \leq_{tc} P/\operatorname{Rad}(P)$. Then $C \leq_{tc} P$ and so there exists a direct summand $K$ of $P$ such that $P = C + K$ and $C \cap K \ll K$. Therefore $C \cap K \leq \operatorname{Rad}(P)$ and

$$P/\operatorname{Rad}(P) = C/\operatorname{Rad}(P) \oplus (K + \operatorname{Rad}(P))/\operatorname{Rad}(P).$$

Hence $P/\operatorname{Rad}(P)$ is $t$-extending.

**Proposition 2.3.** *Let* $R$ *be either a right Noetherian ring or an exchange ring for which* $\operatorname{Rad}(R)$ *is* $Z_2$*-torsion. A finitely generated projective* $R$*-module* $P$ *is* $t$*-supplemented if and only if* $P/\operatorname{Rad}(P)$ *is* $t$*-extending.*

***Proof.*** $(\Rightarrow)$. This follows from Proposition 2.2.

$(\Leftarrow)$. Let $C$ be a $t$-closed submodule of $P$ and $\overline{P}$ denote the factor module $P/\operatorname{Rad}(P)$. Since $\operatorname{Rad}(R)$ is $Z_2$-torsion we conclude that $\operatorname{Rad}(P)$ is $Z_2$-torsion. Thus by [2] (Lemma 2.5(1)), $\operatorname{Rad}(P) \leq C$, and so $\overline{C}$ is a $t$-closed $R$-submodule of $\overline{P}$ by [2] (Proposition 2.6(6)). The $t$-extending property of $\overline{P}$ implies that there exists a decomposition $\overline{P} = \overline{C} \oplus \overline{L}$. Hence $P = C + L$, and $C \cap L = \operatorname{Rad}(P) \ll P$. Let $R$ be right Noetherian. Then $P$ is Noetherian, and so $C$ is finitely generated. Therefore $C \cap L = \operatorname{Rad}(C) \ll C$. Similarly, $C \cap L \ll L$. Thus by [21] (41.14(2)), $P = C \oplus L$, and so $P$ is $t$-supplemented. Now assume that $R$ is an exchange ring. By [9] (11.9), $P$ has the exchange property. Thus by [6] (Theorem 3), there exist submodules $B \leq C$ and $K \leq L$ such that $P = B \oplus K$. Therefore $P = C + K$ and $C \cap K \leq C \cap L \ll P$. This shows that $K$ is a direct summand of $P$ which is a weak supplement, hence a supplement of $C$.

The following results give more relations between the properties of $t$-supplemented and $t$-extending for a projective module.

**Proposition 2.4.** *The following statements are equivalent for a projective module $P$:*

(1) *$P$ is $t$-extending.*

(2) *$P$ is $t$-supplemented and every $t$-closed submodule of $P$ is a supplement.*

*If $\operatorname{Rad}(P) \ll P$, then the above statements are equivalent to*

(3) *$P$ is $t$-supplemented and $C \cap \operatorname{Rad}(P) = \operatorname{Rad}(C) \ll C$ for every $t$-closed submodule $C$.*

***Proof.*** (1) $\Rightarrow$ (2). This is clear by the property of $t$-extending.

(2) $\Rightarrow$ (1). Let $C$ be a $t$-closed submodule of $P$. There exists a direct summand $K$ of $P$ such that $P = C + K$ and $C \cap K \ll K$. By [9] (20.4(9)), $C$ is a supplement of $K$. Therefore by [9] (20.9), $P = C \oplus K$.

(1) $\Rightarrow$ (3). Since each $t$-closed submodule $C$ is a direct summand, $C \cap \operatorname{Rad}(P) = \operatorname{Rad}(C)$ by [9] (20.4(7)). On the other hand, $\operatorname{Rad}(P) \ll P$ implies that $\operatorname{Rad}(C) \ll P$, hence $\operatorname{Rad}(C) \ll C$.

(3) $\Rightarrow$ (1). Let $C$ be a $t$-closed submodule of $P$. There exists a direct summand $K$ of $P$ such that $P = C + K$ and $C \cap K \ll K$. Thus $C \cap K \leq \operatorname{Rad}(K)$ and by hypothesis $C \cap K \leq \operatorname{Rad}(C)$. Now consider the epimorphism $f : C \oplus K \to P$ which is defined by $f(c, k) = c + k$. Since $P$ is projective we conclude that $f$ splits and so $\ker(f)$ is a direct summand of $C \oplus K$. Clearly, $\ker(f) = \{(x, -x) : x \in C \cap K\} \leq \operatorname{Rad}(C) \oplus \operatorname{Rad}(K) = \operatorname{Rad}(C \oplus K)$. But $K$ is a direct summand of $P$ and $\operatorname{Rad}(P) \ll P$, hence $\operatorname{Rad}(K) \ll K$. So $\operatorname{Rad}(C \oplus K) \ll C \oplus K$. Thus $\ker(f) \ll C \oplus K$, and so $\ker(f) = 0$. Hence $P \cong C \oplus K$ which implies that $C$ is a direct summand of $P$.

***Corollary* 2.4.** *Let $P$ be a projective module such that $\operatorname{Rad}(P)$ is $Z_2$-torsion. The following statements are equivalent*:

(1) *$P$ is $t$-extending.*

(2) *$P$ is $t$-supplemented and $Z_2(P)$ is a supplement.*

(3) *$P$ is $t$-supplemented and $Z_2(P)$ is a direct summand of $P$.*

***Proof.*** (1) $\Rightarrow$ (3) $\Rightarrow$ (2). These are obvious.

(2) $\Rightarrow$ (1). Since $\operatorname{Rad}(P)$ is $Z_2$-torsion we conclude that $P/Z_2(P)$ is a homomorphic image of $P/\operatorname{Rad}(P)$. Hence $P/Z_2(P)$ is $t$-extending by Proposition 2.2. Let $C$ be a $t$-closed submodule of $P$. Clearly, $C/Z_2(P)$ is $t$-closed in $P/Z_2(P)$, and so it is a direct summand of $P/Z_2(P)$. Hence by [9] (20.5(2)), $C$ is a supplement in $M$. Thus $P$ is $t$-extending by Proposition 2.4.

***Corollary* 2.5.** *Let $R$ be a right continuous ring. Then the properties of $t$-supplemented and $t$-extending coincide for a projective $R$-module $P$.*

***Proof.*** Since $Z_2(R_R)$ is a direct summand of $R$ we conclude that $Z_2(F)$ is a direct summand of $F$, for every free $R$-module $F$. Thus $Z_2(P)$ is a direct summand of $P$. On the other hand, $\mathrm{Rad}(R) = Z(R_R)$ is $Z_2$-torsion. Hence $\mathrm{Rad}(F)$ is $Z_2$-torsion for every free $R$-module $F$, and this implies that $\mathrm{Rad}(P)$ is $Z_2$-torsion. So by Corollary 2.4, the properties of $t$-supplemented and $t$-extending are equivalent for $P$.

**3. Right (finitely) $\Sigma$-$t$-supplemented rings.** In this section, we show that a ring $R$ whose all (resp., all finitely generated) nonsingular $R$-modules have projective covers is precisely a ring $R$ for which all (resp., all finitely generated) free $R$-modules are $t$-supplemented. Note that a direct sum of $t$-supplemented free modules need not be $t$-supplemented. For example, if $R = \mathbb{Z}[x]$ then by [8] (Example 2.4), $R$ is an extending $R$-module but $R \oplus R$ is not so. For, $R$ being right nonsingular, the properties of extending and $t$-extending are the same and since $\mathrm{Rad}(R) = 0$, the notions of $t$-extending and $t$-supplemented are equivalent by Proposition 2.2. Hence for this ring, a direct sum of $t$-supplemented free modules need not be $t$-supplemented.

**Definition 3.1.** *We say that a ring $R$ is right $\Sigma$-$t$-supplemented if every free $R$-module is $t$-supplemented.*

Recall from [2] that a ring $R$ is right $\Sigma$-$t$-extending if every free $R$-module is $t$-extending. Clearly right $\Sigma$-$t$-extending rings are right $\Sigma$-$t$-supplemented. The next example show that the class of right $\Sigma$-$t$-supplemented rings properly contains the class of right $\Sigma$-$t$-extending rings.

***Example* 3.1.** Let $R = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix}$. Since $\mathrm{Rad}(R) = \begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix}$ is nilpotent and $R/\mathrm{Rad}(R)$ is semisimple, we conclude that $R$ is right perfect and so it is right $\Sigma$-$t$-supplemented. However it is easy to see that $R$ is right nonsingular, hence if it were right $\Sigma$-$t$-extending then $R$ would be right Artinian by [10] (12.21(b)), which is not. Hence $R$ is not right $\Sigma$-$t$-extending.

The following result gives some equivalent conditions for a ring $R$ with the property that all nonsingular $R$-modules have projective covers. The equivalence (1) $\Leftrightarrow$ (4) is in contrast with [2] $\big($Theorem 3.12((1) $\Leftrightarrow$ (2))$\big)$. For brevity let us say that a module $M$ satisfies the property $\mathcal{P}$ if every $t$-closed submodule of $M$ has a supplement in $M$ which has a projective cover.

**Theorem 3.1.** *The following statements are equivalent for a ring $R$:*

(1) *$R$ is right $\Sigma$-$t$-supplemented.*

(2) *Every projective $R$-module is $t$-supplemented.*

(3) *Every $R$-module $M$ satisfies the property $\mathcal{P}$.*

(4) *Every nonsingular $R$-module has a projective cover.*

***Proof.*** (1) $\Rightarrow$ (3). Let $M$ be an $R$-module. There exists a free $R$-module $F$ such that $M \cong F/L$ for some submodule $L$ of $F$. Then it suffices to show that every $t$-closed submodule of $F/L$ has a supplement in $F/L$ which has a projective cover. Assume that $C/L$ is a $t$-closed submodule of $F/L$. Then $C$ is $t$-closed in $F$ by [2] (Proposition 2.6(6)). By Theorem 2.1(4), there exists a submodule $K$ of $F$ such that $K$ has a projective cover, $F = C + K$ and $C \cap K \ll K$. Thus $F/L = C/L + (K+L)/L$ and $C/L \cap (K+L)/L \ll (K+L)/L$. Moreover, $C \cap K \ll K$ implies that $L \cap K \ll K$ and so if $f\colon P \to K$ is a projective cover, then $\pi f\colon P \to (K+L)/L$ is a projective cover where $\pi\colon K \to (K+L)/L$ is the canonical projection. Hence $(K+L)/L$ is a supplement of $C/L$ which has a projective cover.

(3) $\Rightarrow$ (4). Let $M$ be a nonsingular $R$-module. By [2] (Proposition 2.6(6)), the zero submodule is $t$-closed in $M$. Thus by hypothesis, $M$ has a projective cover.

(4) $\Rightarrow$ (2). Let $P$ be a projective $R$-module and $C$ be a $t$-closed submodule of $P$. Then $P/C$ is nonsingular and so it has a projective cover. Hence $P$ is $t$-supplemented by Theorem 2.1(2).

(2) ⇒ (1). This is clear.

***Corollary* 3.1.** *If $R$ is a right $\Sigma$-$t$-supplemented ring, then so is $R/Z_2(R_R)$.*

***Proof.*** Let $\overline{R} = R/Z_2(R_R)$ and $M$ be a nonsingular $\overline{R}$-module. If $m \in Z(M_R)$, then there exists a essential right ideal $I$ of $R$ such that $mI = 0$. By [2] (Proposition 2.2(2)), $\overline{I}$ is an essential right ideal of $\overline{R}$. So $m\overline{I} = 0$ and the nonsingular property of $M_{\overline{R}}$ imply that $m = 0$. Hence $M_R$ is nonsingular. Thus by Theorem 3.1(4), $M_R$ has a projective cover. So by [16] (Lemma 24.15), $M_{\overline{R}}$ has a projective cover. Hence $\overline{R}$ is a right $\Sigma$-$t$-supplemented ring by Theorem 3.1(4).

***Corollary* 3.2.** *Every right perfect ring is right $\Sigma$-$t$-supplemented.*

***Proof.*** This is clear by Theorem 3.1(4).

The next example show that the class of right $\Sigma$-$t$-supplemented rings properly contains the class of right perfect rings. The following lemma is helpful.

**Lemma 3.1.** *If $R = \prod_\Lambda R_\lambda$ where each $R_\lambda$ is a ring, then $Z(R_R) = \prod_\Lambda Z((R_\lambda)_{R_\lambda})$ and $Z_2(R_R) = \prod_\Lambda Z_2\left((R_\lambda)_{R_\lambda}\right)$.*

***Proof.*** It is easy to see that a right ideal $I$ of $R$ is essential if and only if $I$ contains $\bigoplus_\Lambda I_\lambda$ where $I_\lambda$ is an essential right ideal in $(R_\lambda)_{R_\lambda}$. This implies that $Z(R_R) = \prod_\Lambda Z((R_\lambda)_{R_\lambda})$ and so $Z_2(R_R) = \prod_\Lambda Z_2((R_\lambda)_{R_\lambda})$.

***Example* 3.2.** Let $R = \prod_\Lambda \mathbb{Z}/4\mathbb{Z}$, where $\Lambda$ is an infinite set. Since $Z_2(\mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$, Lemma 3.1 implies that $Z_2(R_R) = R$. However $MZ_2(R_R) \leq Z_2(M)$, for every $R$-module $M$. Therefore $Z_2(M) = M$ and so $M$ is $t$-extending. Thus $R$ is right $\Sigma$-$t$-extending hence it is right $\Sigma$-$t$-supplemented. However $\operatorname{Rad}(R) = \prod_\Lambda \operatorname{Rad}(\mathbb{Z}/4\mathbb{Z}) = \prod_\Lambda 2\mathbb{Z}/4\mathbb{Z}$ and so $R/\operatorname{Rad}(R) \cong$ $\cong \prod_\Lambda \mathbb{Z}/2\mathbb{Z}$ is not semisimple. Thus $R$ is not a right perfect ring.

**Proposition 3.1.** *Let $R$ be a right $\Sigma$-$t$-supplemented ring. If $Z_2(R_R)$ is semiprime, then $R/Z_2(R_R)$ is a right hereditary ring.*

***Proof.*** Since $R/Z_2(R_R)$ is a nonsingular $R$-module we conclude that $R/Z_2(R_R)$ is a right nonsingular ring. Hence every free $R/Z_2(R_R)$-module is nonsingular and so is every submodule of a projective $R/Z_2(R_R)$-module. But $R/Z_2(R_R)$ is a right $\Sigma$-$t$-supplemented ring by Corollary 3.1. Therefore by Theorem 3.1(4), submodules of projective $R/Z_2(R_R)$-modules have projective covers. Thus by [11] (Corollary 1.6), $R/Z_2(R_R)$ is a right hereditary ring.

***Corollary* 3.3.** *Let $R$ be a right $\Sigma$-$t$-supplemented right $t$-extending ring. If $Z_2(R_R)$ is semiprime, then $R/Z_2(R_R)$ is a right Noetherian ring.*

***Proof.*** By Proposition 3.1, $R/Z_2(R_R)$ is right hereditary. Moreover, $R/Z_2(R_R)$ is an extending $R$-module by [2] (Theorem 2.11(3)). Hence $R/Z_2(R_R)$ is a right extending ring, and so it is a right Noetherian ring by [10] (Corollary 10.6(1)).

**Proposition 3.2.** *Let $R$ be a right $\Sigma$-$t$-supplemented ring. Then $R$ is a right max ring with the zero radical if and only if $R$ is a right V-ring.*

***Proof.*** Let $R$ be a max ring with the zero radical and $M$ be an $R$-module. By Theorem 3.1(3), $Z_2(M/\operatorname{Rad}(M))$ has a supplement $K/\operatorname{Rad}(M)$ which has a projective cover. Since $\operatorname{Rad}(R) = 0$ we conclude that $K/\operatorname{Rad}(M)$ is projective, and so $\operatorname{Rad}(K) = \operatorname{Rad}(M)$ is a direct summand of $K$. But $R$ is a max ring, and so $\operatorname{Rad}(K) \ll K$. Hence $\operatorname{Rad}(M) = 0$. This implies that $R$ is a right $V$-ring; see [21] (23.1). The converse is clear.

Recall from [3] that $R$ is a right $t$-semisimple ring if $R/Z_2(R_R)$ is semisimple. There, it was shown that $R$ is right $t$-semisimple, if and only if, every nonsingular $R$-module is injective, if and only if, every nonsingular $R$-module is semisimple, if and only if, every nonsingular right ideal of $R$

is a direct summand. The following result shows that for a ring $R$ such that $\mathrm{Rad}(R) \leq Z_2(R_R)$, the properties of right $\Sigma$-$t$-supplemented, right $\Sigma$-$t$-extending, and right $t$-semisimple are equivalent.

**Proposition 3.3.** *The following statements are equivalent for a ring $R$ :*

(1) *$R$ is right $\Sigma$-$t$-supplemented and $\mathrm{Rad}(R)$ is $Z_2$-torsion.*

(2) *$R$ is right $\Sigma$-$t$-extending and $\mathrm{Rad}(R)$ is $Z_2$-torsion.*

(3) *$R$ is right $t$-semisimple.*

***Proof.*** (1) $\Rightarrow$ (3). Let $R^{(\Lambda)}$ be a free $R$-module. By hypothesis, $R^{(\Lambda)}$ is $t$-supplemented and so $R^{(\Lambda)}/\,\mathrm{Rad}(R^{(\Lambda)})$ is $t$-extending by Proposition 2.2. Since $\mathrm{Rad}(R_R)$ is $Z_2$-torsion we conclude that $\mathrm{Rad}(R^{(\Lambda)})$ is $Z_2$-torsion. Thus by [2] (Proposition 2.14(1)), $[R/Z_2(R_R)]^{(\Lambda)} \cong R^{(\Lambda)}/Z_2(R^{(\Lambda)})$ is $t$-extending. But $[R/Z_2(R_R)]^{(\Lambda)}$ is nonsingular, and so $[R/Z_2(R_R)]^{(\Lambda)}$ is extending. This implies that $R/Z_2(R_R)$ is a right $\Sigma$-extending ring. On the other hand, $R/Z_2(R_R)$ is a right nonsingular ring since it is a nonsingular $R$-module. Hence $R/Z_2(R_R)$ is right Artinian by [10] (12.21(b)). Therefore $R$ is right $t$-semisimple by [3] (Corollary 4.4(1)).

(3) $\Rightarrow$ (2). This follows by [3] (Corollary 3.6 and Theorem 2.3(3)).

(2) $\Rightarrow$ (1). This implication is obvious.

***Corollary* 3.4.** *For right continuous rings and rings with zero radical, the properties of right $\Sigma$-$t$-supplemented, right $\Sigma$-$t$-extending, and right $t$-semisimple are equivalent.*

Recall that a ring $R$ is called quasi-Frobenius if $R$ is right or left Artinian and right or left self-injective ring (all cases are equivalent). It is well known that $R$ is quasi-Frobenius if and only if $R$ is left and right self-injective and left or right perfect; see [17] (Theorem 6.39). The Faith conjecture states that every left or right perfect, right self-injective ring $R$ is quasi-Frobenius. This conjecture remains open, but imposing extra condition(s) on $R$ ensures that $R$ is quasi-Frobenius; see [17]. The next result, in particular, shows that a right self-injective right perfect ring with Noetherian or Artinian second singular ideal is exactly a quasi-Frobenius ring.

***Corollary* 3.5.** *The following statements are equivalent*:

(1) *$R$ is a right self-injective right $\Sigma$-$t$-supplemented ring such that $Z_2(R_R)$ is Noetherian.*

(2) *$R$ is a right self-injective right $\Sigma$-$t$-supplemented ring such that $Z_2(R_R)$ is Artinian.*

(3) *$R$ is a quasi-Frobenius ring.*

***Proof.*** (1) $\Rightarrow$ (3). By Corollary 3.4, $R/Z_2(R_R)$ is semisimple. Therefore $R/Z_2(R_R)$ is a Noetherian $R$-module, and so by hypothesis, $R$ is right Noetherian. Thus $R$ is quasi-Frobenius.

Similarly, the implication (2) $\Rightarrow$ (3) can be proved, and clearly, (3) $\Rightarrow$ (1), (2).

**Proposition 3.4.** *The following statements are equivalent for a ring $R$ :*

(1) *$R$ is right $\Sigma$-$t$-supplemented and $\mathrm{Rad}(R) = Z_2(R_R)$.*

(2) *$R$ is right $\Sigma$-$t$-extending and $\mathrm{Rad}(R) = Z_2(R_R)$.*

(3) *$R$ is semisimple.*

***Proof.*** (1) $\Rightarrow$ (3). By Proposition 3.3, $R$ is right $t$-semisimple. So $Z_2(R_R)$ is a direct summand of $R$ by [3] (Theorem 2.3(3)). However, $Z_2(R_R) = \mathrm{Rad}(R)$ implies that $Z_2(R_R) = 0$. Hence $R$ is semisimple.

(3) $\Rightarrow$ (2) $\Rightarrow$ (1). These implications are clear.

***Corollary* 3.6.** *Let $R$ be a von Neumann regular ring. Then $R$ is right $\Sigma$-$t$-supplemented if and only if $R$ is semisimple.*

***Proof.*** This follows from Proposition 3.4.

In the following we consider rings for which every finitely generated (resp., cyclic) nonsingular $R$-module has a projective cover. Let us call a ring $R$ *right finitely $\Sigma$-$t$-supplemented* if every finitely generated free $R$-module is $t$-supplemented.

***Remark* 3.1.** The proof of Theorem 3.1 shows that similar equivalent conditions hold for $R$ if in the statements we replace 'right $\Sigma$-$t$-supplemented' by 'right finitely $\Sigma$-$t$-supplemented', 'right $\Sigma$-$t$-extending' by 'right finitely $\Sigma$-$t$-extending' (see [2], Remark 3.14), and assume that $R$-modules under consideration are finitely generated. So a ring $R$ for which every finitely generated nonsingular $R$-module has a projective cover is precisely a right finitely $\Sigma$-$t$-supplemented ring. Thus every semiperfect ring is right finitely $\Sigma$-$t$-supplemented. However the properties of right finitely $\Sigma$-$t$-supplemented and semiperfect are not equivalent; for example, the ring $R = \prod_{\Lambda} \mathbb{Z}/4\mathbb{Z}$ (for an infinite set $\Lambda$) is right finitely $\Sigma$-$t$-supplemented as shown in Example 3.2, but $R/\operatorname{Rad}(R)$ is not semisimple and so $R$ is not semiperfect.

In Corollary 3.4, the property of right $\Sigma$-$t$-supplemented for right continuous rings and rings with zero radicals is characterized. In the following, we determine when a right continuous ring is right finitely $\Sigma$-$t$-supplemented.

**Theorem 3.2.** *The following statements are equivalent for a right continuous ring $R$ :*

(1) *$R$ is right finitely $\Sigma$-$t$-supplemented.*

(2) *$R$ is right finitely $\Sigma$-$t$-extending.*

(3) *$R/Z_2(R_R)$ is a right self-injective ring.*

(4) *$M/\operatorname{Rad}(M)$ is $t$-extending for every finitely generated* (*free*, *projective*) *$R$-module $M$.*

***Proof.*** The equivalences of (1), (2) follows from Corollary 2.5.

(2) $\Rightarrow$ (3). Clearly, hypothesis implies that $R_\lambda/Z_2((R_\lambda)_{R_\lambda})$ is a right continuous right finitely $\Sigma$-extending ring. So $R_\lambda/Z_2((R_\lambda)_{R_\lambda})$ is a right self-injective ring by [17] (Corollary 7.41((1) $\Leftrightarrow$ (3))). Hence $R/Z_2(R_R)$ is a right self-injective ring.

(3) $\Rightarrow$ (2). Since $R$ is right continuous, $Z_2(R_R)$ is a direct summand of $R$. So $R/Z_2(R_R)$ is a projective $R$-module. Hence by [15] (Corollary 3.6A), $R/Z_2(R_R)$ is an injective $R$-module. Thus $R$ is right finitely $\Sigma$-$t$-extending by [2] (Theorem 2.11(3)).

(1) $\Rightarrow$ (4). Let $M$ be a finitely generated $R$-module. There exists an epimorphism $f: F \to M$ for some finitely generated free $R$-module $F$. Clearly $f(\operatorname{Rad}(F)) \leq \operatorname{Rad}(M)$ and so $\overline{f}: F/\operatorname{Rad}(F) \to M/\operatorname{Rad}(M)$ defined by $\overline{f}(x+\operatorname{Rad}(F)) = f(x)+\operatorname{Rad}(M)$, is an epimorphism. However, $F/\operatorname{Rad}(F)$ is $t$-extending by Proposition 2.2, and so $M/\operatorname{Rad}(M)$ is $t$-extending by [2] (Proposition 2.14(1)).

(4) $\Rightarrow$ (1). Let $F$ be a finitely generated free $R$-module. Since $R$ is right continuous, $R$ is an exchange ring and $\operatorname{Rad}(R)$ is $Z_2$-torsion. Thus by Proposition 2.3, $F$ is $t$-supplemented.

***Corollary* 3.7.** *A von Neumann regular ring $R$ is right finitely $\Sigma$-$t$-supplemented if and only if it is right self-injective.*

***Proof.*** ($\Rightarrow$). By Proposition 2.2, every finitely generated free $R$-module is $t$-extending. So $R$ is right finitely $\Sigma$-$t$-extending. Hence $R$ is right extending as it is right nonsingular. Since $R$ is von Neumann regular we conclude that $R$ is right continuous. Thus by Theorem 3.2(3), $R$ is right self-injective.

($\Leftarrow$). Since every finitely generated free $R$-module is injective, we conclude that $R$ is right finitely $\Sigma$-$t$-extending. So it is right finitely $\Sigma$-$t$-supplemented.

By [10] (18.26), $R$ is quasi-Frobenius, if and only if, $R$ is left and right continuous and left and right Artinian. There are examples of one-sided continuous left and right Artinian rings which are not quasi-Frobenius; see [10] (Examples 18.27). In [1] and [18] one can find more conditions on a right continuous ring to be quasi-Frobenius. The next result, in particular, shows that a right continuous right Artinian ring with injective second singular ideal is exactly a quasi-Frobenius ring.

***Corollary* 3.8.** *The following statements are equivalent*:

(1) *$R$ is a right continuous right finitely $\Sigma$-$t$-supplemented ring such that $Z_2(R_R)$ is injective.*

(2) *$R$ is right self-injective ring.*

***Proof.*** (1) $\Rightarrow$ (2). As shown in the proof of Theorem 3.2 ((3) $\Rightarrow$ (2)), $R/Z_2(R_R)$ is an injective $R$-module. Since $Z_2(R_R)$ is a direct summand of $R$ we conclude that $R$ is right self-injective.

(2) $\Rightarrow$ (1). This is obvious.

***Remark* 3.2.** Recall that every cyclic $R$-module has a projective cover, if and only if, $R$ is a semiperfect ring, if and only if, $R$ is right supplemented. By modifying the proof of Theorem 3.1, similar equivalent conditions hold for $R$ if in the statements we replace 'right $\Sigma$-$t$-supplemented' by 'right $t$-supplemented', 'right $\Sigma$-$t$-extending' by 'right $t$-extending', and assume that $R$-modules under consideration are cyclic. Hence a ring $R$ for which every nonsingular cyclic $R$-module has a projective cover is exactly a right $t$-supplemented ring (which is characterized in Theorem 2.1 and Corollary 2.3).

The next result is in contrast with Corollaries 3.6 and 3.7.

**Proposition 3.5.** *A von Neumann regular ring $R$ is right $t$-supplemented if and only if it is right continuous.*

***Proof.*** Let $R$ be right $t$-supplemented. By Proposition 2.2, $R$ is right $t$-extending. So $R$ is right extending as it is right nonsingular. On the other hand, $R$ has the $C_2$ condition. Thus $R$ is right continuous. The converse implication is clear since every continuous module is $t$-extending by [2] (Theorem 2.11(3)).

Finally we give examples showing that the classes of right $\Sigma$-$t$-supplemented rings, right finitely $\Sigma$-$t$-supplemented rings and right $t$-supplemented rings are indeed different.

***Example* 3.3.** (i) Let $F$ be a field and $F'$ be a proper subfield of $F$. Set $S = \prod_{\mathbb{N}} F$ and assume that $R$ is the subring of $S$ consisting of all $(a_n)_{\mathbb{N}}$ with $a_n \in F'$ for all but a finite number of elements $n \geq 1$. As shown in [10] (Examples 12.20(i)), $R$ is a commutative von Neumann regular ring which is extending but not finitely $\Sigma$-extending. Since every von Neumann regular ring is nonsingular with zero Jacobson radical, Proposition 2.2 shows that $R$ is $t$-supplemented but not finitely $\Sigma$-$t$-supplemented.

(ii) Let $F$ be a field and $V$ be an infinite dimensional vector space over $F$. Then consider the von Neumann regular ring $R = \operatorname{End}(V)$. As shown in [10] (Examples 12.20(ii)), $R$ is right finitely $\Sigma$-extending but not right $\Sigma$-extending. Again, Proposition 2.2 implies that $R$ is right finitely $\Sigma$-$t$-supplemented but not right $\Sigma$-$t$-supplemented.

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