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REFINEMENTS OF JESSEN'S FUNCTIONAL*

УТОЧНЕННЯ ФУНКЦІОНАЛА ЙЄССЕНА

We obtain new refinements of Jessen's functional defined by means of positive linear functionals. The accumulated results are then applied to weighted generalized and power means. We also obtain new refinements of numerous classical inequalities such as the arithmetic-geometric mean inequality, Young's inequality, and Hölder's inequality.

Отримано нові уточнення функціонала Йєссена, визначені у термінах додатних лінійних функціоналів. Отримані результати застосовано до зважених узагальнених та степеневих середніх. Також отримано нові уточнення численних класичних нерівностей, таких як нерівність для арифметично-геометричних середніх, нерівності Янга та Гельдера.

1. Introduction. Let us denote with $\mathcal{P}_n$ the set of all real n-tuples $\mathbf{p} = (p_1, \ldots, p_n)$ such that $P_k := \sum_{i=1}^{k} p_i,\ k = 1, \ldots, n,$ with $0 \leq P_k \leq P_n,\ k = 1, \ldots, n-1,$ and $P_n > 0$. Let I be an interval in $\mathbb{R}$ and $\Phi\colon I \to \mathbb{R}$ a convex function. If $\mathbf{x} = (x_1, \ldots, x_n)$ is a monotonic (increasing or decreasing) n-tuple in I^n and $\mathbf{p}$ is in $\mathcal{P}_n$, then Jensen – Steffensen's inequality (for more details see [17, p. 57])

$$\Phi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \Phi(x_i) \tag{1.1}$$

holds. Now, we define a functional as the difference between the right-hand side and the left-hand side of (1.1) multiplied by P_n

$$J(\Phi, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i \Phi(x_i) - P_n \Phi\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right). \tag{1.2}$$

We call it discrete Jensen – Steffensen's functional. For a fixed function Φ and n-tuple $\mathbf{x}$, $J(\Phi, \mathbf{x}, \cdot)$ can be considered as a function on the set $\mathcal{P}_n$. Because of (1.1) we have that $J(\Phi, \mathbf{x}, \mathbf{p}) \geq 0$ for all $\mathbf{p}$ in $\mathcal{P}_n$.

Inequality (1.1) can be observed under stricter conditions on $\mathbf{p}$ to obtain the well known Jensen's inequality. Let $\Phi\colon I \subset \mathbb{R} \to \mathbb{R}$ be a convex function, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$, $n \geq 2$, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is positive n-tuple of real numbers with $P_n = \sum_{i=1}^{n} p_i$.

In this case, observing the difference between the right-hand side and the left-hand side of Jensen's inequality, Dragomir et al. (see [9]) introduced and investigated discrete Jensen's functional

$$J_n(\Phi, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i \Phi(x_i) - P_n \Phi\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right). \tag{1.3}$$

Let $\mathcal{P}_n^0$ denote the set of all nonnegative n-tuples of real numbers with $P_n = \sum_{i=1}^{n} p_i > 0$. Obviously, $\mathcal{P}_n^0 \subset \mathcal{P}_n$. For a fixed function Φ and n-tuple $\mathbf{x}$, $J_n(\Phi, \mathbf{x}, \cdot)$ can be considered as a function on the set $\mathcal{P}_n^0$.

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Dragomir et al. (see [9]) obtained that such functional is superadditive on the set of positive real n-tuples, that is

$$J_n(\Phi, \mathbf{x}, \mathbf{p}+\mathbf{q}) \geq J_n(\Phi, \mathbf{x}, \mathbf{p}) + J_n(\Phi, \mathbf{x}, \mathbf{q}). \tag{1.4}$$

Further, above functional is also increasing in the same setting, that is,

$$J_n(\Phi, \mathbf{x}, \mathbf{p}) \geq J_n(\Phi, \mathbf{x}, \mathbf{q}) \geq 0, \tag{1.5}$$

where $\mathbf{p} \geq \mathbf{q}$ (i.e., $p_i \geq q_i,\ i = 1, 2, \ldots, n$). Monotonicity property of discrete Jensen's functional was proved few years before (see [13, p. 717]). Above mentioned properties provided refinements of numerous classical inequalities. For more details about such extensions see [9]. Krnić et al. proved the superadditivity property and monotonicity property of the functional (1.1), for more details see [12].

It is well known that Jensen's inequality can be regarded in a more general manner, including positive linear functionals acting on linear class of real valued functions.

More precisely, let E be nonempty set and let $\mathcal{L}(E, \mathbb{R})$ be any linear class of real-valued functions $f\colon E \to \mathbb{R}$ satisfying following properties:

(L_1) $f, g \in \mathcal{L}(E, \mathbb{R}) \Rightarrow \alpha f + \beta g \in \mathcal{L}(E, \mathbb{R})$ for all $\alpha, \beta \in \mathbb{R}$;

(L_2) $1 \in \mathcal{L}(E, \mathbb{R})$, that is, if $f(t) = 1$ for all $t \in E$, then $f \in \mathcal{L}(E, \mathbb{R})$.

We also consider positive linear functionals $A\colon \mathcal{L}(E, \mathbb{R}) \to \mathbb{R}$. That is, we assume that

(A_1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in \mathcal{L}(E, \mathbb{R}),\ \alpha, \beta \in \mathbb{R}$;

(A_2) $f \in \mathcal{L}(E, \mathbb{R}),\ f(t) \geq 0$ for all $t \in E \Rightarrow A(f) \geq 0$.

Further, if

(A_3) $A(1) = 1$

also holds, we say that A is normalized positive linear functional or $A(f)$ is linear mean defined on $\mathcal{L}(E, \mathbb{R})$.

Jessen's generalization of Jensen's inequality (see [17, p. 47, 48]), in view of positive functionals, claims that

$$\Phi(A(f)) \leq A(\Phi(f)), \tag{1.6}$$

where Φ is continuous convex function on interval $I \subseteq \mathbb{R}$, f attains its values on the interval I, A is normalized positive linear functional, and $f \in \mathcal{L}(E, \mathbb{R})$ such that $\Phi(f) \in \mathcal{L}(E, \mathbb{R})$. Jessen's inequality was extensively studied during the eighties and early nineties of the last century (see [7, 8, 10, 14 – 16, 18]).

In this paper we define Jessen's functional including positive functional. Before we define such functional, we have to establish some basic notation see [11].

Let $\mathcal{F}(I, \mathbb{R})$ be the linear space of all real functions on interval $I \subseteq \mathbb{R}$, let $\mathcal{L}(E, \mathbb{R})$ be the linear class of real functions, defined on nonempty set E, satisfying properties (L_1) and (L_2), and let $\mathcal{L}_0^+(E, \mathbb{R}) \subset \mathcal{L}(E, \mathbb{R})$ be subset of nonnegative functions in $\mathcal{L}(E, \mathbb{R})$. Further, let $\mathcal{I}(\mathcal{L}(E, \mathbb{R}), \mathbb{R})$ denotes the space of positive linear functionals on $\mathcal{L}(E, \mathbb{R})$, that is, we assume that such functionals satisfy properties (A_1) and (A_2).

As a generalization of Jensen's functional, with respect to positive functional, we define $J\colon \mathcal{F}(I, \mathbb{R}) \times \mathcal{L}(E, \mathbb{R}) \times \mathcal{L}_0^+(E, \mathbb{R}) \times \mathcal{I}(\mathcal{L}(E, \mathbb{R}), \mathbb{R}) \to \mathbb{R}$ as

$$J(\Phi, f, p; A) = A(p\Phi(f)) - A(p)\Phi\left(\frac{A(pf)}{A(p)}\right). \tag{1.7}$$

Clearly, definition (1.7) is deduced from relation (1.6) and it also contains definition (1.3) of discrete Jensen's functional. We call (1.7) Jessen's functional.

***Remark* 1.1.** In above definition (1.7) we suppose $pf,\ p\Phi(f) \in \mathcal{L}(E,\mathbb{R})$. Then, it is easy to see that $\Phi\left(\frac{A(pf)}{A(p)}\right)$ is well defined provided that $A(p) \neq 0$. Namely, $A_1(f) = \frac{A(pf)}{A(p)} \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$ is normalized positive functional, that is, $A_1(1) = 1$. Suppose $I = [a,b]$. Clearly, $a \leq f(t) \leq b$ for all $t \in E$. Since $f(t) - a \geq 0$, by using properties (A_1), (A_2), and (A_3) we have $A_1(f) - a = A_1(f) - A_1(a) = A_1(f-a) \geq 0$, hence $A_1(f) \geq a$. Similarly, $A_1(f) \leq b$ wherefrom we conclude that $\frac{A(pf)}{A(p)}$ belongs to interval I.

Conditions similar to those in Remark 1.1 will usually be omitted, so Jessen's functional (1.7) will initially assumed to be well defined.

***Remark* 1.2.** If Φ is continuous convex function on interval I, then Jessen's functional is non-negative, i.e.,

$$J(\Phi, f, p; A) \geq 0. \tag{1.8}$$

It follows directly from Jessen's relation (1.6) applied on normalized positive functional

$$A_1(f) = \frac{A(pf)}{A(p)} \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R}).$$

On the other hand, if Φ is continuous concave function, then the sign of inequality in (1.8) is reversed.

Recently, Krnić et al. (see [11]) gave basic properties of Jessen's functional. That proporties are superadditivity and monotonicity. Monotonocity property applies to functions $p, q \in \mathcal{L}_0^+(E,\mathbb{R})$ where $p \geq q$ means $p_i \geq q_i,\ i = 1, 2, \ldots, n$.

Theorem 1.1. *Suppose* $\Phi\colon I \subset \mathbb{R} \to \mathbb{R}$ *is continuous convex function. Let* $f \in \mathcal{L}(E,\mathbb{R})$, $p, q \in \mathcal{L}_0^+(E,\mathbb{R})$, $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$, *such that Jessen's functional* (1.7) *is well defined. Then functional* (1.7) *possess the following properties*:

(i) $J(\Phi, f, \cdot; A)$ *is superadditive on* $\mathcal{L}_0^+(E,\mathbb{R})$, *i.e.*,

$$J(\Phi, f, p+q; A) \geq J(\Phi, f, p; A) + J(\Phi, f, q; A). \tag{1.9}$$

(ii) *If* $p, q \in \mathcal{L}_0^+(E,\mathbb{R})$ *with* $p \geq q$, *then*

$$J(\Phi, f, p; A) \geq J(\Phi, f, q; A) \geq 0, \tag{1.10}$$

i.e., $J(\Phi, f, \cdot; A)$ *is increasing on* $\mathcal{L}_0^+(E,\mathbb{R})$.

(iii) *If* Φ *is continuous concave function, then the signs of inequality in* (1.9) *and* (1.10) *are reversed*, *i.e.*, $J(\Phi, f, \cdot; A)$ *is subadditive and decreasing on* $\mathcal{L}_0^+(E,\mathbb{R})$.

As the first consequence of Theorem 1.1, they obtain monotonicity property of Jessen's functional which includes the function that attains minimum and maximum value on its domain.

***Corollary* 1.1.** *Let Φ be continuous convex function on real interval, $f \in \mathcal{L}(E,\mathbb{R})$, and $A \in \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$. Suppose $p \in \mathcal{L}_0^+(E,\mathbb{R})$ attains minimum and maximum value on the set E. If the functional* (1.7) *is well defined, then the following series of inequalities hold*:

$$\left[\max_{x\in E} p(x)\right] j\left(\Phi, f, 1; A\right) \geq J\left(\Phi, f, p; A\right) \geq \left[\min_{x\in E} p(x)\right] j\left(\Phi, f, 1; A\right), \tag{1.11}$$

where

$$j\left(\Phi, f, 1; A\right) = A\left(\Phi(f)\right) - A(1)\Phi\left(\frac{A(f)}{A(1)}\right). \tag{1.12}$$

Further, if Φ is continuous concave function, then the signs of inequality in (1.11) *are reversed.*

Now we consider the discrete case of Corollary 1.1. We suppose $E = \{1, 2, \ldots, n\}$ and $\mathcal{L}(E,\mathbb{R})$ is the class of real n-tuples. If we consider discrete functional $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$ defined by $A(\mathbf{x}) = \sum_{i=1}^{n} x_i$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then the functional (1.7) becomes discrete functional (1.3) from paper [9] and relation (1.11) takes form

$$\max_{1\leq i\leq n}\{p_i\} S_\Phi(\mathbf{x}) \geq J_n(\Phi, \mathbf{x}, \mathbf{p}) \geq \min_{1\leq i\leq n}\{p_i\} S_\Phi(\mathbf{x}), \tag{1.13}$$

where the functional $J_n(\Phi, \mathbf{x}, \mathbf{p})$ is defined by (1.3) and

$$S_\Phi(\mathbf{x}) = \sum_{i=1}^{n} \Phi(x_i) - n\Phi\left(\frac{\sum_{i=1}^{n} x_i}{n}\right).$$

In this paper we give refinements of Theorem 1.1 and Corollary 1.1.

2. Main results.

Theorem 2.1. *Suppose $\Phi\colon I \subset \mathbb{R} \to \mathbb{R}$ is continuous convex function. Let*

$$f, p, q \in \mathcal{L}(E,\mathbb{R}), \quad A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R}), \quad A(p), A(q) \geq 0, \quad \frac{A(pf)}{A(p)}, \frac{A(qf)}{A(q)} \in I$$

such that Jessen's functional (1.7) *is well defined. Then the following holds*:

(i)
$$\min\{A(p), A(q)\}\left[\Phi\left(\frac{A(pf)}{A(p)}\right) + \Phi\left(\frac{A(qf)}{A(q)}\right) - 2\Phi\left(\frac{A(pf)}{2A(p)} + \frac{A(qf)}{2A(q)}\right)\right] \leq$$
$$\leq J\left(\Phi, f, p+q; A\right) - J\left(\Phi, f, p; A\right) - J\left(\Phi, f, q; A\right) \leq$$
$$\leq \max\{A(p), A(q)\}\left[\Phi\left(\frac{A(pf)}{A(p)}\right) + \Phi\left(\frac{A(qf)}{A(q)}\right) - 2\Phi\left(\frac{A(pf)}{2A(p)} + \frac{A(qf)}{2A(q)}\right)\right]. \tag{2.1}$$

(ii) *If $A(p) \geq A(q) \geq 0$ and $\dfrac{A(pf) - A(qf)}{A(p) - A(q)} \in I$, then*

$$J\left(\Phi, f, p; A\right) - J\left(\Phi, f, q; A\right) \geq$$
$$\geq \min\{A(p) - A(q), A(q)\}\left[\Phi\left(\frac{A(pf) - A(qf)}{A(p) - A(q)}\right) + \Phi\left(\frac{A(qf)}{A(q)}\right) -\right.$$
$$\left. - 2\Phi\left\{\frac{1}{2}\left[\frac{A(pf) - A(qf)}{A(p) - A(q)} + \frac{A(qf)}{A(q)}\right]\right\}\right]. \tag{2.2}$$

Proof. From relation (1.13) in case $n = 2$ we have

$$\min\{\bar{p}, \bar{q}\} \left[\Phi(x) + \Phi(y) - 2\Phi\left(\frac{x+y}{2}\right)\right] \leq$$

$$\leq \bar{p}\Phi(x) + \bar{q}\Phi(y) - (\bar{p}+\bar{q})\Phi\left(\frac{\bar{p}x + \bar{q}y}{\bar{p}+\bar{q}}\right) \leq$$

$$\leq \max\{\bar{p}, \bar{q}\} \left[\Phi(x) + \Phi(y) - 2\Phi\left(\frac{x+y}{2}\right)\right] \tag{2.3}$$

holds. If we supstitute $\bar{p}$ with $A(p)$, $\bar{q}$ with $A(q)$, x with $\dfrac{A(pf)}{A(p)}$ and y with $\dfrac{A(qf)}{A(q)}$ in (2.3) we obtain

$$\min\{A(p), A(q)\} \left[\Phi\left(\frac{A(pf)}{A(p)}\right) + \Phi\left(\frac{A(qf)}{A(q)}\right) - 2\Phi\left(\frac{A(pf)}{2A(p)} + \frac{A(qf)}{2A(q)}\right)\right] \leq$$

$$\leq A(p)\Phi\left(\frac{A(pf)}{A(p)}\right) + A(q)\Phi\left(\frac{A(qf)}{A(q)}\right) - (A(p)+A(q))\Phi\left(\frac{A(pf)+A(qf)}{A(p)+A(q)}\right) \leq$$

$$\leq \max\{A(p), A(q)\} \left[\Phi\left(\frac{A(pf)}{A(p)}\right) + \Phi\left(\frac{A(qf)}{A(q)}\right) - 2\Phi\left(\frac{A(pf)}{2A(p)} + \frac{A(qf)}{2A(q)}\right)\right]. \tag{2.4}$$

From the definition of Jessen's functional (1.7) we get

$$J(\Phi, f, p+q; A) - J(\Phi, f, p; A) - J(\Phi, f, q; A) =$$

$$= A(p)\Phi\left(\frac{A(pf)}{A(p)}\right) + A(q)\Phi\left(\frac{A(qf)}{A(q)}\right) - (A(p)+A(q))\Phi\left(\frac{A(pf)+A(qf)}{A(p)+A(q)}\right). \tag{2.5}$$

So, by combining relations (2.4) and (2.5) we have (2.1).

(ii) Functional $J(\Phi, f, \cdot, A)$ is superadditive and increasing on $\mathcal{L}(E, \mathbb{R})$ and satisfied relation (2.1). So for $A(p) \geq A(q) \geq 0$ and $\dfrac{A(pf) - A(gf)}{A(p) - A(q)} \in I$ the following holds:

$$J(\Phi, f, p; A) - J(\Phi, f, p-q; A) - J(\Phi, f, q; A) \geq$$

$$\geq \min\{A(p-q), A(q)\} \left[\Phi\left(\frac{A((p-q)f)}{A(p-q)}\right) + \Phi\left(\frac{A(qf)}{A(q)}\right) -\right.$$

$$\left. - 2\Phi\left(\frac{A((p-q)f)}{2A(p-q)} + \frac{A(qf)}{2A(q)}\right)\right] \geq$$

$$\geq \min\{A(p) - A(q), A(q)\} \left[\Phi\left(\frac{A(pf) - A(qf)}{A(p) - A(q)}\right) + \Phi\left(\frac{A(qf)}{A(q)}\right) -\right.$$

$$\left. - 2\Phi\left\{\frac{1}{2}\left[\frac{A(pf) - A(qf)}{A(p) - A(q)} + \frac{A(qf)}{A(q)}\right]\right\}\right]. \tag{2.6}$$

Since $J(\Phi, f, p-q; A) \geq 0$ we obtain (2.2).

Theorem 2.1 is proved.

Observe that we can obtain that (2.1) and (2.2) hold also for $p, q \in \mathcal{L}_0^+(E, \mathbb{R})$. That result is given in the following corollary.

***Corollary* 2.1.** *Suppose* $\Phi\colon I \subset \mathbb{R} \to \mathbb{R}$ *is continuous convex function. Let* $f \in \mathcal{L}(E,\mathbb{R})$, $p, q \in \mathcal{L}_0^+(E,\mathbb{R})$, $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$, $A(p), A(q) \geq 0$ *such that Jessen's functional* (1.7) *is well defined. Then the inequality* (2.1) *holds. If* $p \geq q$ *and* $A(p) \geq A(q) \geq 0$, *then* (2.2) *holds.*

Now, we obtain consequence of Corollary 2.1.

***Corollary* 2.2.** *Let* Φ *be continuous convex function on real interval,* $f \in \mathcal{L}(E,\mathbb{R})$, *and* $A \in \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$. *Suppose* $p \in \mathcal{L}_0^+(E,\mathbb{R})$ *attains minimum and maximum value on the set* E. *If the functional* (1.7) *is well defined,* $\underline{p}(x)A(1) \leq A(p) \leq \overline{p}(x)A(1)$, $A(p), A(1) \geq 0$, *then the following series of inequalities hold*:

$$\left[\max_{x\in E} p(x)\right] j\left(\Phi, f, 1; A\right) - J\left(\Phi, f, p; A\right) \geq$$

$$\geq \min\{\overline{p}(x)A(1) - A(p), A(p)\} \left[\Phi\left(\frac{\overline{p}(x)A(f) - A(pf)}{\overline{p}(x)A(1) - A(p)}\right) + \Phi\left(\frac{A(pf)}{A(p)}\right) - \right.$$

$$\left. - 2\Phi\left\{\frac{1}{2}\left[\frac{\overline{p}(x)A(f) - A(pf)}{\overline{p}(x)A(1) - A(p)} + \frac{A(pf)}{A(p)}\right]\right\}\right], \tag{2.7}$$

$$J\left(\Phi, f, p; A\right) - \left[\min_{x\in E} p(x)\right] j\left(\Phi, f, 1; A\right) \geq$$

$$\geq \min\{A(p) - \underline{p}(x)A(1), \underline{p}(x)A(1)\} \left[\Phi\left(\frac{A(pf) - \underline{p}(x)A(f)}{A(p) - \underline{p}(x)A(1)}\right) + \right.$$

$$\left. + \Phi\left(\frac{A(f)}{A(1)}\right) - 2\Phi\left\{\frac{1}{2}\left[\frac{A(pf) - \underline{p}(x)A(f)}{A(p) - \underline{p}(x)A(1)} + \frac{A(f)}{A(1)}\right]\right\}\right], \tag{2.8}$$

where $\overline{p}(x) = \max_{x\in E} p(x)$, $\underline{p}(x) = \min_{x\in E} p(x)$ *and*

$$j\left(\Phi, f, 1; A\right) = A\left(\Phi(f)\right) - A(1)\Phi\left(\frac{A(f)}{A(1)}\right).$$

Proof. Since $p \in \mathcal{L}_0^+(E,\mathbb{R})$ attains minimum and maximum value on its domain E, then

$$\max_{x\in E} p(x) \geq p(x) \geq \min_{x\in E} p(x),$$

so we can consider two constant functions

$$\overline{p}(x) = \max_{x\in E} p(x) \qquad \text{and} \qquad \underline{p}(x) = \min_{x\in E} p(x).$$

Now, double application of property (2.2) yields required result since

$$J\left(\Phi, f, \overline{p}; A\right) = \overline{p}(x) j\left(\Phi, f, 1; A\right) \qquad \text{and} \qquad J\left(\Phi, f, \underline{p}; A\right) = \underline{p}(x) j\left(\Phi, f, 1; A\right).$$

Corollary 2.2 is proved.

***Remark* 2.1.** Let's rewrite relations (2.7) and (2.8) from Corollary 2.2 in a discrete form. We suppose $E = \{1, 2, \ldots, n\}$ and $\mathcal{L}(E,\mathbb{R})$ is the class of real n-tuples. If we consider discrete functional $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$ defined by $A(\mathbf{x}) = \sum_{i=1}^{n} x_i$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then the functional (1.7) becomes discrete functional (1.3) from paper [9] and relation (2.7) takes form

$$\max_{1\le i\le n}\{p_i\}S_\Phi(\mathbf{x}) - J_n(\Phi,\mathbf{x},\mathbf{p}) \ge$$

$$\ge \min\left\{n\max_{1\le i\le n}\{p_i\}-P_n, P_n\right\}\left[\Phi\left(\frac{\max_{1\le i\le n}\{p_i\}\sum_{i=1}^n x_i - \sum_{i=1}^n p_i x_i}{n\max_{1\le i\le n}\{p_i\}-P_n}\right)+\right.$$

$$\left.+\Phi\left(\frac{\sum_{i=1}^n p_i x_i}{P_n}\right) - 2\Phi\left\{\frac{1}{2}\left[\frac{\max_{1\le i\le n}\{p_i\}\sum_{i=1}^n x_i - \sum_{i=1}^n p_i x_i}{n\max_{1\le i\le n}\{p_i\}-P_n} + \frac{\sum_{i=1}^n p_i x_i}{P_n}\right]\right\}\right]$$

and relation (2.8)

$$J_n(\Phi,\mathbf{x},\mathbf{p}) - \min_{1\le i\le n}\{p_i\}S_\Phi(\mathbf{x}) \ge$$

$$\ge \min\left\{P_n - n\min_{1\le i\le n}\{p_i\}, n\min_{1\le i\le n}\{p_i\}\right\}\left[\Phi\left(\frac{\sum_{i=1}^n p_i x_i - \min_{1\le i\le n}\{p_i\}\sum_{i=1}^n x_i}{P_n - n\min_{1\le i\le n}\{p_i\}}\right)+\right.$$

$$\left.+\Phi\left(\frac{\sum_{i=1}^n x_i}{n}\right) - 2\Phi\left\{\frac{1}{2}\left[\frac{\sum_{i=1}^n p_i x_i - \min_{1\le i\le n}\{p_i\}\sum_{i=1}^n x_i}{P_n - n\min_{1\le i\le n}\{p_i\}} + \frac{\sum_{i=1}^n x_i}{n}\right]\right\}\right],$$

where the functional $J_n(\Phi,\mathbf{x},\mathbf{p})$ is defined by (1.3) and

$$S_\Phi(\mathbf{x}) = \sum_{i=1}^n \Phi(x_i) - n\Phi\left(\frac{\sum_{i=1}^n x_i}{n}\right).$$

3. Applications to weighted generalized and power means. In this section we apply our basic results from previous section to weighted generalized and power means with respect to positive functional $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$.

We recall weighted generalized mean with respect to positive linear functional $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$ and continuous and strictly monotone function $\chi \in \mathcal{F}(I,\mathbb{R})$, which is defined as

$$M_\chi(f,p;A) = \chi^{-1}\left(\frac{A(p\chi(f))}{A(p)}\right), \qquad f\in\mathcal{L}(E,\mathbb{R}), \quad p\in\mathcal{L}_0^+(E,\mathbb{R}). \tag{3.1}$$

We assume that (3.1) is well defined, that is, $A(p)\neq 0$ and $p\chi(f)\in\mathcal{L}(E,\mathbb{R})$. Similarly as in the previous section, such conditions will usually be omitted, so weighted generalized mean (3.1) will initially assumed to be well defined.

Now we define functional

$$J_\tau(\chi\circ\psi^{-1},\psi(f),p;A) = A(p)\Big[\chi\left(M_\chi(f,p;A)\right) - \chi\left(M_\psi(f,p;A)\right)\Big] \tag{3.2}$$

where $\psi\colon I\to\mathbb{R}$ is continuous and strictly monotone function such that $\psi(f), p\psi(f)\in\mathcal{L}(E,\mathbb{R})$. It is Jessen's functional (1.7) where the convex function Φ is replaced with $\chi\circ\psi^{-1}$ and $f\in\mathcal{L}(E,\mathbb{R})$ with $\psi(f)\in\mathcal{L}(E,\mathbb{R})$.

Recently Krnić et al. in [11] proved that this functional $J_\tau(\chi\circ\psi^{-1},\psi(f),\cdot;A)$ is superadditive and increasing on $\mathcal{L}_0^+(E,\mathbb{R})$ if $\chi\circ\psi^{-1}$ is a convex function. Now we can generalize their result.

Theorem 3.1. *Let* $\chi, \psi \in \mathcal{F}(I, \mathbb{R})$ *be continuous and strictly monotone functions such that the function* $\chi \circ \psi^{-1}$ *is convex. Suppose* $f \in \mathcal{L}(E, \mathbb{R})$, $p, q \in \mathcal{L}_0^+(E, \mathbb{R})$, $A \in \mathcal{I}(\mathcal{L}(E, \mathbb{R}), \mathbb{R})$, $A(p), A(q) \geq 0$ *are such that the functional* $J_\tau(\chi \circ \psi^{-1}, \psi(f), \cdot\,; A)$ *is well defined. Then, functional* (3.2) *satisfies the following properties*:

(i) $$\min\{A(p), A(q)\} \left[\chi \circ \psi^{-1} \left(\frac{A(p\psi(f))}{A(p)}\right) + \chi \circ \psi^{-1} \left(\frac{A(q\psi(f))}{A(q)}\right) - \right.$$

$$\left. - 2\chi \circ \psi^{-1} \left(\frac{A(p\psi(f))}{2A(p)} + \frac{A(q\psi(f))}{2A(q)}\right)\right] \leq$$

$$\leq J_\tau(\chi \circ \psi^{-1}, \psi(f), p+q; A) - J_\tau(\chi \circ \psi^{-1}, \psi(f), p; A) - J_\tau(\chi \circ \psi^{-1}, \psi(f), q; A) \leq$$

$$\leq \max\{A(p), A(q)\} \left[\chi \circ \psi^{-1} \left(\frac{A(p\psi(f))}{A(p)}\right) + \chi \circ \psi^{-1} \left(\frac{A(q\psi(f))}{A(q)}\right) - \right.$$

$$\left. - 2\chi \circ \psi^{-1} \left(\frac{A(p\psi(f))}{2A(p)} + \frac{A(q\psi(f))}{2A(q)}\right)\right].$$

(ii) *If* $p, q \in \mathcal{L}_0^+(E, \mathbb{R})$ *with* $p \geq q$ *and* $A(p) \geq A(q) \geq 0$, *then*

$$J_\tau(\chi \circ \psi^{-1}, \psi(f), p; A) - J_\tau(\chi \circ \psi^{-1}, \psi(f), q; A) \geq$$

$$\geq \min\{A(p) - A(q), A(q)\}\left[\chi \circ \psi^{-1}\left(\frac{A(p\psi(f)) - A(q\psi(f))}{A(p) - A(q)}\right) + \chi \circ \psi^{-1}\left(\frac{A(q\psi(f))}{A(q)}\right) - \right.$$

$$\left. - 2\chi \circ \psi^{-1} \left\{\frac{1}{2}\left[\frac{A(p\psi(f)) - A(q\psi(f))}{A(p) - A(q)} + \frac{A(q\psi(f))}{A(q)}\right]\right\}\right].$$

Proof. We consider Jessen's functional (1.7) where the convex function Φ is replaced with $\chi \circ \psi^{-1}$ and $f \in \mathcal{L}(E, \mathbb{R})$ with $\psi(f) \in \mathcal{L}(E, \mathbb{R})$. Also functional (3.2) can be rewritten in the form

$$J_\tau\left(\chi \circ \psi^{-1}, \psi(f), p; A\right) = A\left(p \cdot \left(\chi \circ \psi^{-1}\left(\psi(f)\right)\right)\right) - A(p)\chi\left(\psi^{-1}\left(\frac{A\left(p\psi(f)\right)}{A(p)}\right)\right) =$$

$$= A\left(p\chi(f)\right) - A(p)\chi\left(M_\psi(f, p; A)\right) =$$

$$= A(p)\chi\left(M_\chi(f, p; A)\right) - A(p)\chi\left(M_\psi(f, p; A)\right) =$$

$$= A(p)\left[\chi\left(M_\chi(f, p; A)\right) - \chi\left(M_\psi(f, p; A)\right)\right].$$

Now, the properties (i) and (ii) follow from Theorem 1.1.

Theorem 3.1 is proved.

If in Corollary 2.2 we substitute convex function Φ with $\chi \circ \psi^{-1}$ and $f \in \mathcal{L}(E, \mathbb{R})$ with $\psi(f) \in \mathcal{L}(E, \mathbb{R})$ we obtain the following result.

***Corollary* 3.1.** *Suppose* χ, ψ, f, A *are defined as in Theorem* 3.1 *and* $p \in \mathcal{L}_0^+(E, \mathbb{R})$ *attains minimum and maximum value on the set* E. *If the function* $\chi \circ \psi^{-1}$ *is convex,* $\underline{p}(x)A(1) \leq A(p) \leq \leq \overline{p}(x)A(1)$, $A(p), A(1) \geq 0$, *then for the functional* $J_\tau(\chi \circ \psi^{-1}, \psi(f), \cdot\,; A)$ *defined by* (3.2) *the following series of inequalities hold*:

$$\left[\max_{x \in E} p(x)\right] J_\tau\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right) - J_\tau\left(\chi \circ \psi^{-1}, \psi(f), p; A\right) \geq$$

$$\geq \min\{\overline{p}(x)A(1)-A(p),A(p)\}\left[\chi\circ\psi^{-1}\left(\frac{\overline{p}(x)A(\psi(f))-A(p\psi(f))}{\overline{p}(x)A(1)-A(p)}\right)+\right.$$

$$\left.+\chi\circ\psi^{-1}\left(\frac{A(p\psi(f))}{A(p)}\right)-2\chi\circ\psi^{-1}\left\{\frac{1}{2}\left[\frac{\overline{p}(x)A(\psi(f))-A(p\psi(f))}{\overline{p}(x)A(1)-A(p)}+\frac{A(p\psi(f))}{A(p)}\right]\right\}\right],$$

$$J_\tau\left(\chi\circ\psi^{-1},\psi(f),p;A\right)-\left[\min_{x\in E}p(x)\right]J_\tau\left(\chi\circ\psi^{-1},\psi(f),1;A\right)\geq$$

$$\geq \min\{A(p)-\underline{p}(x)A(1),\underline{p}(x)A(1)\}\left[\chi\circ\psi^{-1}\left(\frac{A(p\psi(f))-\underline{p}(x)A(\psi(f))}{A(p)-\underline{p}(x)A(1)}\right)+\right.$$

$$\left.+\chi\circ\psi^{-1}\left(\frac{A(\psi(f))}{A(1)}\right)-2\chi\circ\psi^{-1}\left\{\frac{1}{2}\left[\frac{A(p\psi(f))-\underline{p}(x)A(\psi(f))}{A(p)-\underline{p}(x)A(1)}+\frac{A(\psi(f))}{A(1)}\right]\right\}\right],$$

where $\overline{p}(x)=\max_{x\in E}p(x)$, $\underline{p}(x)=\min_{x\in E}p(x)$,

$$J_\tau\left(\chi\circ\psi^{-1},\psi(f),1;A\right)=A(1)\left[\chi\left(M_\chi(f;A)\right)-\chi\left(M_\psi(f;A)\right)\right]$$

and

$$M_\eta\left(f;A\right)=\eta^{-1}\left(\frac{A\left(\eta(f)\right)}{A(1)}\right),\quad \eta=\chi,\psi.$$

Let $r\in\mathbb{R}$ and $f\in\mathcal{L}_0^+(E,\mathbb{R})$ such that $f(x)>0$ for all $x\in E$. Generalized power mean $M^{[r]}\left(f,p;A\right)$ equipped with positive functional $A\in\mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$ is defined by

$$M^{[r]}\left(f,p;A\right)=\begin{cases}\left(\dfrac{A\left(pf^r\right)}{A(p)}\right)^{1/r}, & r\neq 0,\\[2ex] \exp\left(\dfrac{A\left(p\ln(f)\right)}{A(p)}\right), & r=0,\end{cases}\tag{3.3}$$

where $p\in\mathcal{L}_0^+(E,\mathbb{R})$. We assume that the above expression is well defined, that is, $pf^r\in\mathcal{L}_0^+(E,\mathbb{R})$, $p\ln(f)\in\mathcal{L}(E,\mathbb{R})$, and $A(p)\neq 0$.

Let now $r,s\in\mathbb{R}, s\neq 0$. We define a functional

$$J_P(\chi\circ\psi^{-1},\psi(f),p;A)=A(p)\left\{\left[M^{[s]}(f,p;A)\right]^s-\left[M^{[r]}(f,p;A)\right]^s\right\},\tag{3.4}$$

where $\chi,\psi\colon I\to\mathbb{R}$ are functions defined by $\chi(x)=x^s, s\neq 0$, $\psi(x)=x^r, r\neq 0$ and $\psi(x)=$ $=\ln x, r=0$. The first consequence of Theorem 3.1 refers to generalized power means $M^{[r]}\left(f,p;A\right)$, $r\in\mathbb{R}$. Results are given in the following corollary.

***Corollary* 3.2.** *Let* $s\neq 0$ *and* r *be real numbers,* $f,p,q\in\mathcal{L}_0^+(E,\mathbb{R})$, $f(x)>0$ *for all* $x\in E$, *and* $A\in\mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$, $A(p),A(q)\geq 0$. *The functional* (3.4) *has the following properties*:

(i) *If* $r\neq 0$ *and* $s>0, s>r$ *or* $s<0, s<r$, *then*

$$\min\{A(p),A(q)\}\left[\left(\frac{A\left(pf^r\right)}{A(p)}\right)^{s/r}+\left(\frac{A\left(qf^r\right)}{A(q)}\right)^{s/r}-2\left(\frac{A(pf^r)}{2A(p)}+\frac{A(qf^r)}{2A(q)}\right)^{s/r}\right]\leq$$

$$\leq J_P(\chi\circ\psi^{-1},\psi(f),p+q;A)-J_P(\chi\circ\psi^{-1},\psi(f),p;A)-J_P(\chi\circ\psi^{-1},\psi(f),q;A)\leq$$

$$\leq \max\{A(p), A(q)\}\left[\left(\frac{A(pf^r)}{A(p)}\right)^{s/r} + \left(\frac{A(qf^r)}{A(q)}\right)^{s/r} - 2\left(\frac{A(pf^r)}{2A(p)} + \frac{A(qf^r)}{2A(q)}\right)^{s/r}\right].$$

(ii) *If* $r \neq 0$ *and* $s > 0,\ s > r$ *or* $s < 0,\ s < r$, *then for* $p, q \in \mathcal{L}_0^+(E,\mathbb{R})$ *with* $p \geq q$ *and* $A(p) \geq A(q) \geq 0$, *holds inequality*

$$J_P(\chi\circ\psi^{-1}, \psi(f), p; A) - J_P(\chi\circ\psi^{-1}, \psi(f), q; A) \geq$$

$$\geq \min\{A(p-q), A(q)\}\left[\left(\frac{A(pf^r) - A(qf^r)}{A(p) - A(q)}\right)^{s/r} + \left(\frac{A(qf^r)}{A(q)}\right)^{s/r} -\right.$$

$$\left. - 2^{1-\frac{s}{r}}\left(\frac{A(pf^r) - A(qf^r)}{A(p) - A(q)} + \frac{A(qf^r)}{A(q)}\right)^{s/r}\right].$$

(iii) *If* $r = 0$, *then*

$$\min\{A(p), A(q)\}\left[\exp\left(\frac{sA(p\ln f)}{A(p)}\right) + \exp\left(\frac{sA(q\ln f)}{A(q)}\right) -\right.$$

$$\left. - 2\exp\left(\frac{sA(p\ln f)}{2A(p)} + \frac{sA(q\ln f)}{2A(q)}\right)\right] \leq$$

$$\leq J_P(\chi\circ\psi^{-1}, \psi(f), p+q; A) - J_P(\chi\circ\psi^{-1}, \psi(f), p; A) - J_P(\chi\circ\psi^{-1}, \psi(f), q; A) \leq$$

$$\leq \max\{A(p), A(q)\}\left[\exp\left(\frac{sA(p\ln f)}{A(p)}\right) + \exp\left(\frac{sA(q\ln f)}{A(q)}\right) -\right.$$

$$\left. - 2\exp\left(\frac{sA(p\ln f)}{2A(p)} + \frac{sA(q\ln f)}{2A(q)}\right)\right].$$

(iv) *If* $r = 0$, *then for* $p, q \in \mathcal{L}_0^+(E,\mathbb{R})$ *with* $p \geq q$ *and* $A(p) \geq A(q) \geq 0$, *holds inequality*

$$J_P(\chi\circ\psi^{-1}, \psi(f), p; A) - J_P(\chi\circ\psi^{-1}, \psi(f), q; A) \geq$$

$$\geq \min\{A(p-q), A(q)\}\left[\exp\left(\frac{sA(p\ln f) - sA(q\ln f)}{A(p) - A(q)}\right) + \exp\left(\frac{sA(q\ln f)}{A(q)}\right) -\right.$$

$$\left. - 2\exp\left\{\frac{s}{2}\left[\frac{A(p\ln f) - A(q\ln f)}{A(p) - A(q)} + \frac{A(q\ln f)}{A(q)}\right]\right\}\right].$$

Proof. The proof is direct use of Theorem 3.1. We have to consider two cases depending on whether $r \neq 0$ or $r = 0$.

If $r \neq 0$, we define $\chi(x) = x^s$ and $\psi(x) = x^r$. Then $\chi\circ\psi^{-1}(x) = x^{s/r}$ and $\left(\chi\circ\psi^{-1}\right)''(x) = \frac{s(s-r)}{r^2}x^{s/r-2}$. Thus, $\chi\circ\psi^{-1}$ is convex if $s > 0, s > r$ or $s < 0, s < r$. On the other hand, $\chi\circ\psi^{-1}$ is concave if $s > 0, s < r$ or $s < 0, s > r$.

If $r = 0$, we put $\chi(x) = x^s$ and $\psi(x) = \ln x$. Then, $\chi\circ\psi^{-1}(x) = e^{sx}$ is convex under assumption $s \neq 0$. Results follow immediately from Theorem 3.1.

Corollary 3.2 is proved.

***Corollary* 3.3.** *Suppose* $s \neq 0$ *and* r *be real numbers such that* $r \neq 0,\ s > 0, s > r$ *or* $s < 0$, $s < r$ *and* $p \in \mathcal{L}_0^+(E, \mathbb{R})$ *attains minimum and maximum value on the set* E. *If* $\underline{p}(x)A(1) \leq$ $\leq A(p) \leq \overline{p}(x)A(1),\ A(p), A(1) \geq 0$, *then for the functional* $J_P(\chi \circ \psi^{-1}, \psi(f), \cdot; A)$ *defined by* (3.4) *the following series of inequalities hold*:

$$\left[\max_{x\in E} p(x)\right] J_P\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right) - J_P\left(\chi \circ \psi^{-1}, \psi(f), p; A\right) \geq$$

$$\geq \min\{\overline{p}(x)A(1) - A(p), A(p)\} \left[\left(\frac{\overline{p}(x)A(f^r) - A(pf^r)}{\overline{p}(x)A(1) - A(p)}\right)^{s/r} + \right.$$

$$\left. + \left(\frac{A(pf^r)}{A(p)}\right)^{s/r} - 2^{1-s/r}\left(\frac{\overline{p}(x)A(f^r) - A(pf^r)}{\overline{p}(x)A(1) - A(p)} + \frac{A(pf^r)}{A(p)}\right)^{s/r}\right], \tag{3.5}$$

$$J_P\left(\chi \circ \psi^{-1}, \psi(f), p; A\right) - \left[\min_{x\in E} p(x)\right] J_P\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right) \geq$$

$$\geq \min\{A(p) - \underline{p}(x)A(1), \underline{p}(x)A(1)\} \left[\left(\frac{A(pf^r) - \underline{p}(x)A(f^r)}{A(p) - \underline{p}(x)A(1)}\right)^{s/r} + \right.$$

$$\left. + \left(\frac{A(f^r)}{A(1)}\right)^{s/r} - 2^{1-s/r}\left(\frac{A(pf^r) - \underline{p}(x)A(f^r)}{A(p) - \underline{p}(x)A(1)} + \frac{A(f^r)}{A(1)}\right)^{s/r}\right]. \tag{3.6}$$

If $r = 0$, *then*

$$\left[\max_{x\in E} p(x)\right] J_P\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right) - J_P\left(\chi \circ \psi^{-1}, \psi(f), p; A\right) \geq$$

$$\geq \min\{\overline{p}(x)A(1) - A(p), A(p)\} \left[\exp\left(s\frac{\overline{p}(x)A(\ln f) - A(p\ln f)}{\overline{p}(x)A(1) - A(p)}\right) + \right.$$

$$\left. + \exp\left(\frac{sA(p\ln f)}{A(p)}\right) - 2\exp\left\{\frac{s}{2}\left[\frac{\overline{p}(x)A(\ln f) - A(p\ln f)}{\overline{p}(x)A(1) - A(p)} + \frac{A(p\ln f)}{A(p)}\right]\right\}\right], \tag{3.7}$$

$$J_P\left(\chi \circ \psi^{-1}, \psi(f), p; A\right) - \left[\min_{x\in E} p(x)\right] J_P\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right) \geq$$

$$\geq \min\{A(p) - \underline{p}(x)A(1), \underline{p}(x)A(1)\} \left[\exp\left(s\frac{A(p\ln f) - \underline{p}(x)A(\ln f)}{A(p) - \underline{p}(x)A(1)}\right) + \right.$$

$$\left. + \exp\left(\frac{sA(\ln f)}{A(1)}\right) - 2\exp\left\{\frac{s}{2}\left[\frac{A(p\ln f) - \underline{p}(x)A(\ln f)}{A(p) - \underline{p}(x)A(1)} + \frac{A(\ln f)}{A(1)}\right]\right\}\right], \tag{3.8}$$

where $\overline{p}(x) = \max_{x\in E} p(x),\ \underline{p}(x) = \min_{x\in E} p(x)$,

$$J_P\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right) = A(1)\left\{\left[M^{[s]}(f; A)\right]^s - \left[M^{[r]}(f; A)\right]^s\right\}$$

and

$$M^{[t]}(f;A)=\begin{cases}\left(\dfrac{A\left(f^t\right)}{A(1)}\right)^{1/t}, & t\neq 0,\\ \exp\left(\dfrac{A\left(\ln(f)\right)}{A(1)}\right), & t=0,\end{cases}\qquad t=r,s. \tag{3.9}$$

Now, we consider a discrete variant of relations (3.5)–(3.8). As in Remark 2.1, we suppose $E=\{1,2,\ldots,n\}$, $n\in\mathbb{N}$, and $\mathcal{L}(E,\mathbb{R})$ is a class of real n-tuples. We consider discrete functional $A\in\mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$ defined by $A(\mathbf{x})=\sum_{i=1}^n x_i$, where $\mathbf{x}=(x_1,x_2,\ldots,x_n)$. Clearly, $A(\mathbf{1})==\sum_{i=1}^n 1=n$.

Generalized power mean (3.3) in discrete case becomes

$$M_r(\mathbf{x},\mathbf{p})=\begin{cases}\left(\dfrac{\sum_{i=1}^n p_i x_i^r}{P_n}\right)^{1/r}, & r\neq 0,\\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{1/P_n}, & r=0,\end{cases}$$

where $x_i,p_i\geq 0$, $i=1,\ldots,n$. For $r=1$ we obtain arithmetic mean $A_n(\mathbf{x},\mathbf{p})=M_1(\mathbf{x},\mathbf{p})==\left(\dfrac{1}{P_n}\sum_{i=1}^n p_i x_i\right)$, while for $r=0$ geometric mean $G_n(\mathbf{x},\mathbf{p})=M_0(\mathbf{x},\mathbf{p})=\left(\prod_{i=1}^n x_i^{p_i}\right)^{1/P_n}$. Now, if we take constant n-tuples

$$\overline{\mathbf{p}}=\left(\max_{1\leq i\leq n}\{p_i\},\ldots,\max_{1\leq i\leq n}\{p_i\}\right)\qquad\text{or}\qquad\underline{\mathbf{p}}=\left(\min_{1\leq i\leq n}\{p_i\},\ldots,\min_{1\leq i\leq n}\{p_i\}\right)$$

expression for arithemetic and geometric mean reduce to

$$A_n^0(\mathbf{x})=\frac{1}{n}\sum_{i=1}^n x_i,\qquad\text{and}\qquad G_n^0(\mathbf{x})=\left(\prod_{i=1}^n x_i\right)^{1/n}$$

and inequalities (3.7) and (3.8) for $s=1$ and $r=0$ can be rewritten as

$$n\max_{1\leq i\leq n}\{p_i\}\left[A_n^0(\mathbf{x})-G_n^0(\mathbf{x})\right]-P_n\left[A_n(\mathbf{x},\mathbf{p})-G_n(\mathbf{x},\mathbf{p})\right]\geq$$
$$\geq\min\{n\max_{1\leq i\leq n}\{p_i\}-P_n,P_n\}\times$$
$$\times\left[\exp\left(\frac{\max_{1\leq i\leq n}\{p_i\}\ln(G_n^0(\mathbf{x}))^n-\ln(G_n(\mathbf{x},\mathbf{p}))^{P_n}}{n\max_{1\leq i\leq n}\{p_i\}-P_n}\right)+G_n(\mathbf{x},\mathbf{p})-\right.$$
$$\left.-2\exp\left\{\frac{1}{2}\left[\frac{\max_{1\leq i\leq n}\{p_i\}\ln(G_n^0(\mathbf{x}))^n-\ln(G_n(\mathbf{x},\mathbf{p}))^{P_n}}{n\max_{1\leq i\leq n}\{p_i\}-P_n}+\ln G_n(\mathbf{x},\mathbf{p})\right]\right\}\right], \tag{3.10}$$

$$P_n\left[A_n(\mathbf{x},\mathbf{p})-G_n(\mathbf{x},\mathbf{p})\right]-n\min_{1\leq i\leq n}\{p_i\}\left[A_n^0(\mathbf{x})-G_n^0(\mathbf{x})\right]\geq$$
$$\geq\min\{P_n-n\min_{1\leq i\leq n}\{p_i\},n\min_{1\leq i\leq n}\{p_i\}\}\times$$
$$\times\left[\exp\left(\frac{\ln(G_n(\mathbf{x},\mathbf{p}))^{P_n}-\min_{1\leq i\leq n}\{p_i\}\ln(G_n^0(\mathbf{x}))^n}{P_n-n\min_{1\leq i\leq n}\{p_i\}}\right)+G_n^0(\mathbf{x})-\right.$$

$$- 2\exp\left\{\frac{1}{2}\left[\frac{\ln(G_n(\mathbf{x},\mathbf{p}))^{P_n} - \min_{1\le i\le n}\{p_i\}\ln(G_n^0(\mathbf{x}))^n}{P_n - n\min_{1\le i\le n}\{p_i\}} + \ln G_0(\mathbf{x})\right]\right\}\Bigg]. \qquad (3.11)$$

Some variants of inequalities (3.10) and (3.11) were studied in papers [1 – 6].

***Remark* 3.1.** Young's inequality follows directly from arithmetic-geometric mean inequality, so relations (3.10) and (3.11) provide refinements of Young's inequality. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be positive n-tuples such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. We denote

$$\mathbf{x}^{\mathbf{p}} = (x_1^{p_1}, x_2^{p_2}, \ldots, x_n^{p_n}) \qquad \text{and} \qquad \mathbf{p}^{-1} = \left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right).$$

Then series of inequalities (3.10) and (3.11) can be rewritten in the form

$$n\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[A_n^0(\mathbf{x}^{\mathbf{p}}) - G_n^0(\mathbf{x}^{\mathbf{p}})\right] - \left[A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\right] \ge$$

$$\ge \min\left\{n\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} - 1, 1\right\}\times$$

$$\times\left[\exp\left(\frac{\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\ln(G_n^0(\mathbf{x}^{\mathbf{p}}))^n - \ln G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})}{n\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} - 1}\right) + G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) -\right.$$

$$\left.- 2\exp\left\{\frac{1}{2}\left[\frac{\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\ln(G_n^0(\mathbf{x}^{\mathbf{p}}))^n - \ln G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})}{n\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} - 1} + \ln G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\right]\right\}\right], \qquad (3.12)$$

$$A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[A_n^0(\mathbf{x}^{\mathbf{p}}) - G_n^0(\mathbf{x}^{\mathbf{p}})\right] \ge$$

$$\ge \min\left\{1 - n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}, n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\right\}\times$$

$$\times\left[\exp\left(\frac{\ln G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - \min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\ln(G_n^0(\mathbf{x}^{\mathbf{p}}))^n}{1 - n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}}\right) + G_n^0(\mathbf{x}^{\mathbf{p}}) -\right.$$

$$\left.- 2\exp\left\{\frac{1}{2}\left[\frac{\ln G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - \min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\ln(G_n^0(\mathbf{x}^{\mathbf{p}}))^n}{1 - n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} + \ln(G_0(\mathbf{x}^{\mathbf{p}}))\right]\right\}\right]. \qquad (3.13)$$

Note that Corollaries 3.2 and 3.3 do not cover the case when $s = 0$ and $r \neq 0$. This case should be considered separately. Let $r \neq 0$ be real number, $f, p, q \in \mathcal{L}_0^+(E, \mathbb{R})$, $f(x) > 0$ for all $x \in E$, and $A \in \mathcal{I}(\mathcal{L}(E, \mathbb{R}), \mathbb{R})$. Then we define a functional

$$J_{\overline{P}}(\chi \circ \psi^{-1}, \psi(f), p; A) = A(p) \left\{ \frac{A\left(p \ln f\right)}{A(p)} - \ln \left[M^{[r]}(f, p; A) \right] \right\}, \tag{3.14}$$

where $\chi(x) = \ln x$ and $\psi(x) = x^r$.

***Corollary* 3.4.** *Let $r < 0$ be real number, let $f, p, q \in \mathcal{L}_0^+(E, \mathbb{R})$, $f(x) > 0$ for all $x \in E$ and $A \in \mathcal{I}(\mathcal{L}(E, \mathbb{R}), \mathbb{R})$, $A(p), A(q) \geq 0$. The functional* (3.14) *satisfies the following*:

(i) $$\min\{A(p), A(q)\} \frac{1}{r} \left[\ln\left(\frac{A(pf^r)}{A(p)} \right) + \ln\left(\frac{A(qf^r)}{A(q)} \right) - 2 \ln\left(\frac{A(pf^r)}{2A(p)} + \frac{A(qf^r)}{2A(q)} \right) \right] \leq$$

$$\leq J_{\overline{P}}(\chi \circ \psi^{-1}, \psi(f), p + q; A) - J_{\overline{P}}(\chi \circ \psi^{-1}, \psi(f), p; A) - J_{\overline{P}}(\chi \circ \psi^{-1}, \psi(f), q; A) \leq$$

$$\leq \max\{A(p), A(q)\} \frac{1}{r} \left[\ln\left(\frac{A(pf^r)}{A(p)} \right) + \ln\left(\frac{A(qf^r)}{A(q)} \right) - 2 \ln\left(\frac{A(pf^r)}{2A(p)} + \frac{A(qf^r)}{2A(q)} \right) \right].$$

(ii) *If $p, q \in \mathcal{L}_0^+(E, \mathbb{R})$ with $p \geq q$ and $A(p) \geq A(q) \geq 0$, then*

$$J_{\overline{P}}(\chi \circ \psi^{-1}, \psi(f), p; A) - J_{\overline{P}}(\chi \circ \psi^{-1}, \psi(f), q; A) \geq$$

$$\geq \min\{A(p) - A(q), A(q)\} \frac{1}{r} \left[\ln\left(\frac{A(pf^r) - A(qf^r)}{A(p) - A(q)} \right) + \ln\left(\frac{A(qf^r)}{A(q)} \right) - \right.$$

$$\left. - 2 \ln\left\{ \frac{1}{2} \left[\frac{A(pf^r) - A(qf^r)}{A(p) - A(q)} + \frac{A(qf^r)}{A(q)} \right] \right\} \right].$$

Proof. The proof is direct consequence of Theorem 3.1. We define $\chi(x) = \ln x$ and $\psi(x) = x^r$. Then, the function $\chi \circ \psi^{-1}(x) = \dfrac{1}{r} \ln x$ is convex if $r < 0$ and concave if $r > 0$.

Corollary 3.4 is proved.

The analogue of Corollary 3.3, that covers the case when $s = 0$ and $r \neq 0$, is contained in the following result.

***Corollary* 3.5.** *Let $r < 0$ be real number, $f \in \mathcal{L}_0^+(E, \mathbb{R})$, $f(x) > 0$ for all $x \in E$, $A \in \in \mathcal{I}(\mathcal{L}(E, \mathbb{R}), \mathbb{R})$ and $p \in \mathcal{L}_0^+(E, \mathbb{R})$ attains minimum and maximum value on its domain E. Assume that functional* (3.14) *is well defined. If $\underline{p}(x)A(1) \leq A(p) \leq \overline{p}(x)A(1)$, $A(p), A(1) \geq 0$, then for the functional $J_{\overline{P}}(\chi \circ \psi^{-1}, \psi(f), \cdot; A)$ defined by* (3.14) *the following series of inequalities hold*:

$$\left[\max_{x \in E} p(x) \right] J_{\overline{P}}\left(\chi \circ \psi^{-1}, \psi(f), 1; A \right) - J_{\overline{P}}\left(\chi \circ \psi^{-1}, \psi(f), p; A \right) \geq$$

$$\geq \min\{\overline{p}(x)A(1) - A(p), A(p)\} \frac{1}{r} \left[\ln\left(\frac{\overline{p}(x)A(f^r) - A(pf^r)}{\overline{p}(x)A(1) - A(p)} \right) + \right.$$

$$\left. + \ln\left(\frac{A(pf^r)}{A(p)} \right) - 2 \ln\left\{ \frac{1}{2} \left[\frac{\overline{p}(x)A(f^r) - A(pf^r)}{\overline{p}(x)A(1) - A(p)} + \frac{A(f^r)}{A(p)} \right] \right\} \right],$$

$$J_{\overline{P}}\left(\chi \circ \psi^{-1}, \psi(f), p; A \right) - \left[\min_{x \in E} p(x) \right] J_{\overline{P}}\left(\chi \circ \psi^{-1}, \psi(f), 1; A \right) \geq$$

$$\geq \min\{A(p) - \underline{p}(x)A(1), \underline{p}(x)A(1)\}\frac{1}{r}\left[\ln\left(\frac{A(pf^r) - \underline{p}(x)A(f^r)}{A(p) - \underline{p}(x)A(1)}\right) + \right.$$

$$\left. + \ln\left(\frac{A(f^r)}{A(1)}\right) - 2\ln\left\{\frac{1}{2}\left[\frac{A(pf^r) - \underline{p}(x)A(f^r)}{A(p) - \underline{p}(x)A(1)} + \frac{A(f^r)}{A(1)}\right]\right\}\right],$$

where $\overline{p}(x) = \max_{x\in E} p(x),\ \underline{p}(x) = \min_{x\in E} p(x),\ M^{[r]}(f;A)$ *is defined by* (3.9) *and*

$$J_{\overline{P}}\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right) = A(1)\left(\frac{A(\ln f)}{A(1)} - \ln\left[M^{[r]}(f;A)\right]\right).$$

4. Applications to Hölder's inequality. This section is devoted to Hölder's inequality. In view of positive functional $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$, Hölder's inequality claims that

$$A\left(\prod_{i=1}^{n} f_i^{1/p_i}\right) \leq \prod_{i=1}^{n} A^{1/p_i}(f_i),$$

where $p_i,\ i = 1, 2, \ldots, n$, are conjugate exponents, that is $\sum_{i=1}^{n} 1/p_i = 1,\ p_i > 1,\ i = 1, 2, \ldots, n$, and provided that $f_1, f_2, \ldots, f_n,\ \prod_{i=1}^{n} f_i^{1/p_i} \in \mathcal{L}_0^+(E,\mathbb{R})$.

It is well known from the literature (see [13, 17]) that Hölder's inequality can easily be obtained from Young's inequality. If we consider n-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where $x_i = [f_i/A(f_i)]^{1/p_i}$, $i = 1, 2, \ldots, n$, the expressions in (3.12) and (3.13), that represent the difference between arithmetic and geometric mean, become

$$A_n(\mathbf{x}^{\mathbf{P}}, \mathbf{p}^{-1}) - G_n(\mathbf{x}^{\mathbf{P}}, \mathbf{p}^{-1}) = \sum_{i=1}^{n} \frac{f_i}{p_i A(f_i)} - \prod_{i=1}^{n} \frac{f_i^{1/p_i}}{A^{1/p_i}(f_i)},$$

$$A_n^0(\mathbf{x}^{\mathbf{P}}) - G_n^0(\mathbf{x}^{\mathbf{P}}) = \frac{1}{n}\sum_{i=1}^{n} \frac{f_i}{A(f_i)} - \prod_{i=1}^{n} \frac{f_i^{1/n}}{A^{1/n}(f_i)}.$$

Now, if we apply positive functional $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$ on above expressions, and use its linearity property, we get

$$A\left[A_n(\mathbf{x}^{\mathbf{P}}, \mathbf{p}^{-1}) - G_n(\mathbf{x}^{\mathbf{P}}, \mathbf{p}^{-1})\right] = \sum_{i=1}^{n} \frac{A(f_i)}{p_i A(f_i)} - \frac{A\left(\prod_{i=1}^{n} f_i^{1/p_i}\right)}{\prod_{i=1}^{n} A^{1/p_i}(f_i)} =$$

$$= 1 - \frac{A\left(\prod_{i=1}^{n} f_i^{1/p_i}\right)}{\prod_{i=1}^{n} A^{1/p_i}(f_i)},$$

and

$$A\left[A_n^0(\mathbf{x}^{\mathbf{P}}) - G_n^0(\mathbf{x}^{\mathbf{P}})\right] = \frac{1}{n}\sum_{i=1}^{n} \frac{A(f_i)}{A(f_i)} - \frac{A\left(\prod_{i=1}^{n} f_i^{1/n}\right)}{\prod_{i=1}^{n} A^{\frac{1}{n}}(f_i)} =$$

$$= 1 - \frac{A\left(\prod_{i=1}^{n} {f_i}^{1/n}\right)}{\prod_{i=1}^{n} A^{1/n}(f_i)}.$$

By application of functional $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$ on the series of inequalities in (3.12) and (3.13), the signs of inequalities do not change, since A is linear and positive. Here we will give only result involving inequality (3.12). Analogous result can be obtained for inequality (3.13), but here we omit the details.

Theorem 4.1. *Let* $p_i > 1$, $i = 1, 2, \ldots, n$, *be conjugate exponents,* $f_i \in \mathcal{L}_0^+(E,\mathbb{R})$, $i = 1, 2, \ldots, n$, *and* $\prod_{i=1}^{n} {f_i}^{1/p_i}$, $\prod_{i=1}^{n} {f_i}^{1/n} \in \mathcal{L}_0^+(E,\mathbb{R})$. *If* $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$, *then the following series of inequalities hold*:

$$n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[\prod_{i=1}^{n} A^{1/p_i}(f_i) - \prod_{i=1}^{n} A^{1/p_i - 1/n}(f_i)\, A\left(\prod_{i=1}^{n} {f_i}^{1/n}\right) - \right.$$

$$\left. - \prod_{i=1}^{n} A^{1/p_i}(f_i) - A\left(\prod_{i=1}^{n} {f_i}^{1/p_i}\right)\right] \ge$$

$$\ge \min\{n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} - 1, 1\} \prod_{i=1}^{n} A^{1/p_i}(f_i) \times$$

$$\times \left[A\left\{\exp\left(\frac{n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} \ln \prod_{i=1}^{n} \frac{{f_i}^{1/n}}{A^{1/n}(f_i)} - \ln \prod_{i=1}^{n} \frac{{f_i}^{1/p_i}}{A^{1/p_i}(f_i)}}{n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} - 1}\right)\right\} + A\left(\prod_{i=1}^{n} {f_i}^{1/p_i}\right) - \right.$$

$$\left. - 2A\left\{\exp\left(\frac{n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} \ln \prod_{i=1}^{n} \frac{{f_i}^{1/n}}{A^{1/n}(f_i)} - \ln \prod_{i=1}^{n} \frac{{f_i}^{1/p_i}}{A^{1/p_i}(f_i)}}{2\left(n \max\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\} - 1\right)} + \frac{1}{2} \ln \prod_{i=1}^{n} \frac{{f_i}^{1/p_i}}{A^{1/p_i}(f_i)}\right)\right\}\right].$$

It is also well known that Hölder's inequality can directly be deduced from Jensen's inequality in the case of two functions (see [13]). Let $r, s \in \mathbb{R}$ such that $1/r + 1/s = 1$. Let $f, g \in \mathcal{L}_0^+(E,\mathbb{R})$ and $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$. We define a functional

$$J_H\left(\Phi, \frac{g}{f}, f; A\right) = rs\left[A^{1/r}(f) A^{1/s}(g) - A\left(f^{1/r} g^{1/s}\right)\right],$$

where $\Phi\colon I \to \mathbb{R}$ is defined by $\Phi(x) = -rsx^{1/s}$. It is Jessen's functional (1.7) where the convex function Φ is replaced with $\Phi(x) = -rsx^{1/s}$ and arguments f and p respectively replaced with g/f and f.

We obain the following result.

Theorem 4.2. *Let* $1/r + 1/s = 1$, *with* $r > 1$, *let* $f, g \in \mathcal{L}_0^+(E,\mathbb{R})$, *and* $A \in \mathcal{I}(\mathcal{L}(E,\mathbb{R}),\mathbb{R})$. *If the function* f *attains minimum and maximum value on set* E, *then the following series of inequalities hold*:

$$\left[\max_{x\in E} f(x)\right]\left[A^{1/r}(1)A^{1/s}\left(\frac{g}{f}\right)-A\left(\left(\frac{g}{f}\right)^{1/s}\right)\right]-$$

$$-A^{1/r}(f)A^{1/s}(g)-A\left(f^{1/r}g^{1/s}\right)\geq$$

$$\geq \min\left\{\left[\max_{x\in E} f(x)\right]A(1)-A(f),A(f)\right\}\times$$

$$\times\left[2^{1-1/s}\left(\frac{\left[\max_{x\in E} f(x)\right]A\left(\frac{g}{f}\right)-A(g)}{\left[\max_{x\in E} f(x)\right]A(1)-A(f)}+\frac{A(g)}{A(f)}\right)^{1/s}-\right.$$

$$\left.-\left(\frac{\left[\max_{x\in E} f(x)\right]A\left(\frac{g}{f}\right)-A(g)}{\left[\max_{x\in E} f(x)\right]A(1)-A(f)}\right)^{1/s}+A^{1/r}(f)A^{1/s}(g)\right]. \tag{4.1}$$

Proof. We consider relation (2.7) from Corollary 2.2 with arguments f and p respectively replaced with g/f and f, where $\Phi(x)=-rsx^{1/s}$. Clearly, $\Phi''(x)=x^{1/s-2}$, so Φ is convex function if $x>0$. In this setting, Jessen's functional (1.7) reads

$$J_H\left(\Phi,\frac{g}{f},f;A\right)=A\left(f\Phi\left(\frac{g}{f}\right)\right)-A(f)\Phi\left(\frac{A(g)}{A(f)}\right)=$$

$$=rs\left[A^{1-1/s}(f)A^{1/s}(g)-A\left(f^{1-1/s}g^{1/s}\right)\right]=$$

$$=rs\left[A^{1/r}(f)A^{1/s}(g)-A\left(f^{1/r}g^{1/s}\right)\right].$$

Further,

$$J_H\left(\Phi,\frac{g}{f},1;A\right)=A\left(\Phi\left(\frac{g}{f}\right)\right)-A(1)\Phi\left(\frac{A\left(\frac{g}{f}\right)}{A(1)}\right)=$$

$$=rs\left[A^{1-1/s}(1)A^{1/s}\left(\frac{g}{f}\right)-A\left(\left(\frac{g}{f}\right)^{1/s}\right)\right]=$$

$$=rs\left[A^{1/r}(1)A^{1/s}\left(\frac{g}{f}\right)-A\left(\left(\frac{g}{f}\right)^{1/s}\right)\right].$$

Now, we substitute obtained expressions $J_H\left(\Phi,g/f,f;A\right)$ and $J_H\left(\Phi,g/f,1;A\right)$ in (2.7) and obtain (4.1).

Theorem 4.2 is proved.

***Remark* 4.1.** We can also consider relation (2.8) from Corollary 2.2 to obtain analogous result to (4.1), but here we omit the details.

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