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ON THE INTERFERENCE OF THE WEIGHT AND BOUNDARY CONTOUR FOR ALGEBRAIC POLYNOMIALS IN WEIGHTED LEBESGUE SPACES. II*

ПРО ВЗАЄМНИЙ ВПЛИВ ВАГИ ТА ГРАНИЧНОГО КОНТУРА ДЛЯ АЛГЕБРАЇЧНИХ ПОЛІНОМІВ У ВАГОВИХ ПРОСТОРАХ ЛЕБЕГА. II

We continue to study the estimation of the modulus of algebraic polynomials on the boundary contour with weight function, when the contour and the weight function have certain singularities with respect to the their quasinorm in the weighted Lebesgue space. In particular, the exact estimates were obtained for polynomials orthonormal on the curve with respect to the weight function with zeros on the same curve.

Ми продовжуємо дослідження модулів алгебраїчних поліномів на граничному контурі з ваговою функцією у випадку, коли цей контур та вагова функція мають деякі сингулярності відносно їх квазінорми у ваговому просторі Лебега. Зокрема, точні оцінки було отримано для поліномів, ортонормальних на кривій відносно вагової функції з нулями на цій кривій.

1. Introduction. Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region, with $0 \in G$ and the boundary $L := \partial G$ is a closed Jordan curve, $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = \operatorname{ext} L$. Let $\wp_n$ denotes the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N} := \{1, 2, \ldots\}$.

Let $0 < p \leq \infty$. For a rectifiable Jordan curve L, we denote

$$\|P_n\|_{\mathcal{L}_p} := \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int\limits_L h(z) \, |P_n(z)|^p \, |dz| \right)^{1/p}, \quad 0 < p < \infty,$$

$$\|P_n\|_{\mathcal{L}_\infty} := \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \quad p = \infty.$$

Clearly, $\|\cdot\|_{\mathcal{L}_p}$ is a quasinorm (i.e., a norm for $1 \leq p \leq \infty$ and a p-norm for $0 < p < 1$).

Denoted by $w = \Phi(z)$ the univalent conformal mapping of Ω onto $\Delta := \{w \colon |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \dfrac{\Phi(z)}{z} > 0$ and $\Psi := \Phi^{-1}$. For $t \geq 1$ we set

$$L_t := \{z \colon |\Phi(z)| = t\}, \quad L_1 \equiv L, \ G_t := \operatorname{int} L_t, \quad \Omega_t := \operatorname{ext} L_t.$$

For some fixed R_0, $1 < R_0 < \infty$ and $z \in G_{R_0}$, we consider the so-called generalized Jacobi weight function $h(z)$, is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \tag{1.1}$$

where $\gamma_j > -1$, for all $j = 1, 2, \ldots, m$ and h_0 is uniformly separated from zero in G_{R_0}, i.e., there exists a constant $c_0 := c_0(G_{R_0}) > 0$ such that for all $z \in G_{R_0}$

$$h_0(z) \geq c_0 > 0.$$

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We will continue our study which is started in [22] and [23], was about how the inequalities for algebraic polynomials change depending on weight function and common properties of the curve. The general shape of these inequalities was given as follows:

$$\|P_n\|_{\mathcal{L}_q(h,L)} \le c\mu_n(L,h,p,q)\,\|P_n\|_{\mathcal{L}_p(h,L)}\,, \quad 0<p<q\le\infty, \tag{1.2}$$

where $c=c(L,p,q)>0$ is a constant independent of n and P_n, and $\mu_n(L,h,p,q)\to\infty,\ n\to\infty$ depending on the geometrical properties of curve L and weight function h in the neighborhood of the points $\{z_j\}_{j=1}^m$.

Just to remind, in general, these types of inequalities are common in literature. First results of (1.2)-type, in case $h(z)\equiv 1$ and $L=\{z\colon |z|=1\}$ for $0<p<\infty$, was found in [15] and, for the sufficiently smooth curve, was obtained in [28] ($h(z)\equiv 1$) and [30] (Part 4) ($h(z)\neq 1$). The estimation of (1.2)-type for $0<p<\infty$ and $h(z)\equiv 1$, where L is a rectifiable Jordan curve, that was investigated in [18, 19, 21, p. 122–133, 26, 29] and for $h(z)\neq 1$ in [13] (Theorem 6), [5–9] and others. Other related results, regarding the inequality of (1.2)-type, can be obtained from references cited above and in Milovanovic et al. [20] (Sect. 5.3).

Let a rectifiable Jordan curve L has a natural parametrization $z=z(s), 0\le s\le l:=\operatorname{mes} L$. By the following [31], it is said that $L\in C(1,\lambda),\ 0<\lambda<1$, if $z(s)$ is continuously differentiable and $z'(s)\in \operatorname{Lip}\lambda$. Let L belongs to $C(1,\lambda)$ everywhere except for a single point $z_1\in L$, i.e., the derivative $z'(s)$ satisfies the Lipschitz condition on the $[0,l]$ and $z(0)=z(l)=z_1$, however $z'(0)\neq z'(l)$. We assume that L has a corner at z_1 with exterior angle $\nu_1\pi,\ 0<\nu_1\le 2$. It is denoted the set of such curves by $C(1,\lambda,\nu_1)$.

In [31] (Theorem 1), Suetin investigated this problem for $p=2,\ q=\infty$ and orthonormal on the curve $L\in C(1,\lambda,\nu_1)$ polynomials $K_n(z)$ with the weight function h defined as in (1.1), in cases $(1+\gamma_1)\,\nu_1=1$ and $(1+\gamma_1)\,\nu_1\neq 1$. In particular, he showed, if the singularity of a curve and weight function at the point z_1 satisfies the following condition:

$$(1+\gamma_1)\,\nu_1>1, \tag{1.3}$$

then, for $|K_n(z)|$, the following is true:

$$\begin{gathered} |z-z_1|^{\mu}\,|K_n(z)|\le c_1(L)\sqrt{n}, \quad z\in L,\\ |K_n(z_1)|\le c_2(L)n^{\sigma}, \quad z\in L, \end{gathered} \tag{1.4}$$

where

$$\mu:=\frac{1}{2}\left(\gamma_1+1-\frac{1}{\nu_1}\right), \quad \sigma:=\frac{1}{2}\,(1+\gamma_1)\,\nu_1, \tag{1.5}$$

and $c_i(L)>0,\ i=1,2$, are the constants independent on n and z.

In this work we studied the estimations of the (1.4)-type, for more general curves of the complex plane and we obtained the analog of the inequalities (1.3) and (1.4), corresponding to the general case.

2. Definitions and main results. Throughout this paper, $c,\ c_0,\ c_1,\ c_2, \ldots$ are positive and $\varepsilon_0,$ $\varepsilon_1,\ \varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations), which depend on G in general and on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

Let $z = \psi(w)$ be the univalent conformal mapping of $B := \{w: \ |w| < 1\}$ onto the G normalized by $\psi(0) = 0,\ \psi'(0) > 0.$

By [24, p. 286–294], we say, a bounded Jordan region G is called a κ-quasidisk, $0 \leq \kappa < 1,$ if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \dfrac{1+\kappa}{1-\kappa}$, homeomorphism of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$. In that case, the curve $L := \partial G$ is called a κ-quasicircle. The region G (curve L) is called a *quasidisk (quasicircle)*, if it is κ-quasidisk (κ-quasicircle) for some $0 \leq \kappa < 1.$

We denoted the class of κ-quasicircle by $Q(\kappa),\ 0 \leq \kappa < 1,$ and said $L \in Q,$ if $L \in Q(\kappa),$ for some $0 \leq \kappa < 1.$

It is well-known that the quasicircle may not even be locally rectifiable by [16, p. 104].

Definition 2.1. *We say that* $L \in \widetilde{Q},\ 0 \leq \kappa < 1,$ *if L is a quasicircle and rectifiable.*

Definition 2.2. *We say that* $L \in Q_\alpha,\ 0 < \alpha \leq 1,$ *if L is a quasicircle and* $\Phi \in \operatorname{Lip}\alpha,\ z \in \overline{\Omega}.$

It is noted, the class Q_α is sufficiently wide. A detailed account on it and the related topics are contained in [17, 25, 32] (see also the references cited therein). We consider only some cases.

***Remark* 2.1.** 1. If $L = \partial G$ is a Dini-smooth curve [25, p. 48], then $L \in Q_1.$

2. If $L = \partial G$ is a piecewise Dini-smooth curve and largest exterior angle at L has opening $\alpha\pi, 0 < \alpha \leq 1,$ in [25, p. 52], then $L \in Q_\alpha.$

3. If $L = \partial G$ is a smooth curve having continuous tangent line, then $L \in Q_\alpha$ for all $0 < \alpha < 1.$

4. If L is quasismooth (in the sense of Lavrentiev), for every pair $z_1,\ z_2 \in L,$ if $s(z_1, z_2)$ represents the smallest of the lengths of the arcs joining z_1 to z_2 on $L,$ there exists a constant $c(L) > 1$ such that $s(z_1, z_2) \leq c\,|z_1 - z_2|\,,$ then $\Phi \in \operatorname{Lip}\alpha$ for $\alpha = \dfrac{1}{2}\left(1 - \dfrac{1}{\pi}\arcsin\dfrac{1}{c}\right)^{-1},$ see [32].

5. If L is "c-quasiconformal" (see, for example, [17]), then $\Phi \in \operatorname{Lip}\alpha$ for $\alpha = \dfrac{\pi}{2\left(\pi - \arcsin\dfrac{1}{c}\right)}.$

Also, if L is an asymptotic conformal curve, then $\Phi \in \operatorname{Lip}\alpha$ for all $0 < \alpha < 1,$ see [17].

Definition 2.3. *We say that* $L \in \widetilde{Q}_\alpha,\ 0 < \alpha \leq 1,$ *if* $L \in Q_\alpha$ *and L is rectifiable.*

In everywhere in the future, notation $i = \overline{k, m}$ means $i = k, k+1, \ldots, m,$ for any $k \geq 0$ and $m > k.$

Theorem A. *Let* $p > 0.$ *Suppose that* $L \in \widetilde{Q}_\alpha,$ *for some* $0 < \alpha \leq 1$ *and $h(z)$ is defined as in* (1.1). *Then, for any* $\gamma_i > -1,\ i = \overline{1, m},$ *and* $P_n \in \wp_n,\ n \in \mathbb{N},$ *there exists* $c_3 = c_3(L, p, \gamma_i, \alpha) > 0$ *such that*

$$|P_n(z_i)| \leq c_3 \left\|P_n\right\|_{\mathcal{L}_p(h,L)} \begin{cases} n^{\frac{\gamma_i+1}{\alpha p}}, & \dfrac{1}{2} \leq \alpha \leq 1, \\ n^{\frac{\delta(\gamma_i+1)}{p}}, & 0 < \alpha < \dfrac{1}{2}, \end{cases} \tag{2.1}$$

where $\delta = \delta(L),\ \delta \in [1, 2]\,,$ *is a certain number.*

Therefore, according to (2.1), we can calculate α in the right parts of the estimation (2.1) for each case, respectively.

Lets introduce special "singular" points on the curve L. For this goal lets give the following definition. For $\delta > 0$ and $z \in \mathbb{C};$ we set $B(z, \delta) := \{\zeta: \ |\zeta - z| < \delta\}\,,\ \Omega(z, \delta) := \Omega \cap\ B(z, \delta).$

Definition 2.4 [2]. *We say that* $L \in Q_{\alpha,\beta_1,\ldots,\beta_m}, 0 < \beta_i \le \alpha \le 1,\ i = \overline{1,m}$, *if*

i) *for every sequence non-crossing in pairs circles* $\{B(\zeta_i,\delta_i)\}_{i=1}^m$ *restriction of the function* Φ *on* $\Omega(\zeta_i,\delta_i)$ *which belongs to* $\operatorname{Lip}\beta_i$ $(\Phi\,|\Omega(\zeta_i,\delta_i) \in \operatorname{Lip}\beta_i)$, *and restriction*

$$\Phi\left|\ \Omega\backslash\bigcup_{i=1}^{m}\Omega(\zeta_i,\delta_i) \in \operatorname{Lip}\alpha;\right.$$

ii) *there exists a sequence non-crossing in pairs circles* $\{B(\zeta_i,\delta_i^*)\}_{i=1}^m$, *such that for all* $i = \overline{1,m}$, $\delta_i^* > \delta_i$ *and* $\xi, z \in \Omega(\zeta_i,\delta_i^*)$, $z \neq \zeta_i \neq \xi$ *is fulfilled estimation*

$$|\Phi(z) - \Phi(\xi)| \le k_i(z,\xi)\,|z-\xi|^{\alpha}, \tag{2.2}$$

where

$$k_i(z,\xi) = c_i \max\left(|\xi-\zeta_i|^{\beta_i-\alpha}; |z-\zeta_i|^{\beta_i-\alpha}\right),$$

and c_i *does not depend on* z *and* ξ.

Definition 2.5. *We say that* $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}, 0 < \beta_i \le \alpha \le 1, i = \overline{1,m}$, *if* $L \in Q_{\alpha,\beta_1,\ldots,\beta_m}$, $0 < \beta_i \le \alpha \le 1,\ i = \overline{1,m}$, *and* $L = \partial G$ *is rectifiable.*

It is clear from the Definitions 2.4 and 2.5, that is each region $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}, 0 < \beta_i \le \alpha \le 1$, $i = \overline{1,m}$, may have “singularity” at the points $\{\zeta_i\}_{i=1}^m \in L$ on $\alpha - \beta_i$ order. If the curve L does not have such “singularity”, i.e., if $\beta_i = \alpha$, for all $i = \overline{1,m}$, then it is written as $L \in \widetilde{Q}_{\alpha}, 0 < \alpha \le 1$.

Throughout this work, we will assume that the points $\{z_i\}_{i=1}^m \in L$ are defined in (1.1) and $\{\zeta_i\}_{i=1}^m \in L$ are defined in Definitions 2.4 and 2.5 coincides. Without loss of generality, we will also assume that the points $\{z_i\}_{i=1}^m$ are ordered in the positive direction on the curve L. In [22], we showed the following result:

Theorem B [22]. *Let* $p > 0$. *Suppose that* $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}$, *for some* $\dfrac{1}{2} \le \beta_i \le \alpha \le 1$, $i = \overline{1,m}$, *and* $h(z)$ *is defined in* (1.1), *and*

$$\gamma_i + 1 = \frac{\beta_i}{\alpha} \tag{2.3}$$

for each points $\{z_i\}_{i=1}^m$. *Then, for any* $P_n \in \wp_n,\ n \in \mathbb{N}$, *there exists* $c_4 = c_4(L,p,\gamma_i,\alpha) > 0$ *such that*

$$\|P_n\|_{\mathcal{L}_\infty} \le c_4 n^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p(h,L)}. \tag{2.4}$$

Condition (2.3) is called the condition of interference of singularity at the points $\{z_i\}_{i=1}^m$. From Theorems B and A (with $\gamma_i = 0$, for all $i = \overline{1,m}$) we see that, under the conditions (2.3) at all critical points $\{z_i\}_{i=1}^m \in L$, the presence of singularity does not affect the estimate of the rate of growth of the polynomials $P_n(z)$ on L.

In the present work, we investigate the case such that $\gamma_i + 1 > \dfrac{\beta_i}{\alpha}$ for each singular points $\{z_i\}_{i=1}^m \in L$ and we obtained the following main results:

Theorem 2.1. *Let* $p > 0$. *Suppose that* $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}$, *for some* $\frac{1}{2} \leq \beta_i \leq \alpha \leq 1,\ i = \overline{1,m}$, *and* $h(z)$ *is defined in* (1.1), *and*

$$\gamma_i + 1 > \frac{\beta_i}{\alpha} \tag{2.5}$$

for each points $\{z_i\}_{i=1}^m$. *Then there exists* $c_j = c_j(L, p, \gamma_i, \beta_i, \alpha) > 0,\ j = 5, 6$, *such that, for any* $P_n \in \wp_n,\ n \in \mathbb{N}$, *we have*

$$\max_{z \in L} \left(\prod_{i=1}^m |z - z_i|^{\mu_i} |P_n(z)| \right) \leq c_5 n^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.6}$$

$$|P_n(z_1)| \leq c_6 n^{s_1} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.7}$$

where

$$\mu_i := \frac{1}{p}\left(\gamma_i + 1 - \frac{\beta_i}{\alpha}\right), \quad s_i = \frac{\gamma_i + 1}{p\beta_i}, \quad i = \overline{1,m}. \tag{2.8}$$

It follows from the conditions $\frac{1}{2} \leq \beta_i \leq \alpha \leq 1,\ i = \overline{1,m}$, the conditions (2.5) will be satisfied where $\gamma_i > 0,\ i = \overline{1,m}$. For that reason, we will call (2.5) algebraic zero conditions of the order $\lambda_i := \frac{\alpha}{\beta_i}(1 + \gamma_i) - 1$ on each singular point on L.

For the curve $L \in C(1, \lambda, \nu_1)$, in case of one singular point on L, we have the following corollary.

***Corollary* 2.1.** *If* $L \in C(1, \lambda, \nu_1)$, *then* $L \in \widetilde{Q}_{\alpha,\beta_1}$ *for* $\alpha = 1$ (2.1) *and* $\beta_1 = \frac{1}{\nu_1}$ *by* [17]. *Consequently, if the condition*

$$(\gamma_1 + 1)\,\nu_1 > 1,$$

is satisfies at the point z_1, *then for* $p = 2$ *from* (2.5) *and* (2.6), *we have*

$$|z - z_1|^{\mu_1} |P_n(z)| \leq c_7 \sqrt{n}\, \|P_n\|_{\mathcal{L}_2(h,L)}, \tag{2.9}$$

$$|P_n(z_1)| \leq c_8 n^{s_1} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.10}$$

where

$$\mu_1 := \frac{1}{2}\left(\gamma_1 + 1 - \frac{1}{\nu_1}\right), \quad s_1 = \frac{1}{2}(1 + \gamma_1)\,\nu_1. \tag{2.11}$$

For the $P_n \equiv K_n$, i.e., orthonormal polynomials $K_n(z)$ on the contour $L \in C(1, \lambda, \nu_1)$ with the same weight function h, estimation (2.9) coincides from the result by P. K. Suetin [31] (Theorem 3). Therefore, Theorem 2.1 generalizes the result [31] (Theorem 3) for $1 \leq \nu_1 \leq 2$ and extends the result to more general curves of the complex plane.

Moreover, according to the [10] (Theorem 2.6), we have the following corollary.

***Corollary* 2.2.** *Let* $L \in C(1, \lambda)$; $h(z)$ *is defined in* (1.1) *with* $\gamma_i > 0,\ i = \overline{1,m}$, *and suppose that there exists a point* $z_{i_0} \in \{z_i\}_{i=1}^m$ *such that*

$$\|K_n\|_{\mathcal{L}_\infty} \asymp |K_n(z_{i_0})|.$$

Then

$$|K_n(z_{i_0})| \preceq (n+1)^{s_{i_0}-\frac{1}{2}}, \tag{2.12}$$

where

$$s_{i_0} = \frac{\gamma_{i_0}+1}{2}. \tag{2.13}$$

***Example* 2.1.** Let $L^* := \{z\colon |z| = 1\}$ and $h^*(z) = |z-1|^2$. In this case, the orthonormal polynomials $K_n^*(z)$ along the contour L^* and weight function h^* will be written as follows:

$$K_n^*(z) = \frac{1}{\sqrt{(n+1)(n+2)}}\left[1 + 2z + \ldots + (n+1)z^n\right].$$

Then we have

$$\|K_n^*\|_{\mathcal{L}_\infty} = |K_n^*(1)| = \frac{\sqrt{(n+1)(n+2)}}{2} \asymp n. \tag{2.14}$$

The estimation (2.14) shown that, generally speaking, the exponent $s_{i_0} - \dfrac{1}{2}$ and s_{i_0} in inequalities (2.12) and (2.13) cannot be replaced by smaller numbers.

***Remark* 2.2.** 1. The inequalities (2.1), (2.7) are sharp. For the polynomials $P_n^*(z) = 1 + z + \ldots \ldots + z^n$, a) $h^*(z) \equiv 1$, b) $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$, and $L := \{z\colon |z| = 1\}$, there exists a constants $c_9 = c_9(p) > 0$ and $c_{10} = c_{10}(h^{**}, p) > 0$ such that

a) $\|P_n^*\|_{\mathcal{L}_\infty} \geq c_9 n^{\frac{1}{p}} \|P_n^*\|_{\mathcal{L}_p(1,\, L)}$, $p > 1$;

b) $\|P_n^*\|_{\mathcal{L}_\infty} \geq c_{10} n^{\frac{\gamma+1}{p}} \|P_n^*\|_{\mathcal{L}_p(h^{**},\, L)}$, $p > \gamma + 1$.

2. The inequalities (2.6) and (2.9) are sharp in the sense that for the arbitrary polynomial $P_n \in \wp_n$, $L \in \widetilde{Q}_{\alpha,\beta_1}$ and for arbitrary ε, $0 < \varepsilon < \mu_1$, the following is true:

$$|z - z_1|^{\mu_1 - \varepsilon} |P_n(z)| \leq c_{11} n^{\frac{1}{p\alpha}+\varepsilon} \|P_n\|_{\mathcal{L}_p(h,L)},$$

where

$$\mu_1 := \frac{1}{p}\left(\gamma_1 + 1 - \frac{\beta_1}{\alpha}\right).$$

In particular, for each ε^*, $0 < \varepsilon^* < \mu_1^*$, there exists a contour L such that

$$|z - z_1|^{\mu_1^* - \varepsilon^*} |P_n(z)| \leq c_{12} n^{\frac{1}{2}+\varepsilon^*} \|P_n\|_{\mathcal{L}_2(h,L)},$$

where

$$\mu_1^* := \frac{1}{2}\left(\gamma_1 + 1 - \frac{1}{\nu_1}\right).$$

We note that, case of $\gamma_i + 1 < \dfrac{\beta_i}{\alpha}$, $i = \overline{1,m}$, was investigated in [23]. Similar results for integral over an area are obtained in [3, 4].

3. Some auxiliary results. For $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \le cb$ and "$a \asymp b$" are equivalent to $c_1 a \le b \le c_2 a$ for some constants $c,\ c_1,\ c_2$ (independent of a and b) respectively.

The following definitions of K-quasiconformal curves are well-known (see, for example, [11; 16, p. 97; 27]).

Definition 3.1. *The Jordan arc* (*or curve*) L *is called* K*-quasiconformal* ($K \ge 1$), *if there is a* K*-quasiconformal mapping* f *of the region* $D \supset L$ *such that* $f(L)$ *is a line segment* (*or circle*).

Let $F(L)$ *denotes the set of all sense preserving plane homeomorphisms* f *of the region* $D \supset L$ *such that* $f(L)$ *is a line segment* (*or circle*) *and let defines*

$$K_L := \inf \{K(f) : f \in F(L)\},$$

where $K(f)$ *is the maximal dilatation of a such mapping* f. L *is a quasiconformal curve, if* $K_L < \infty$, *and* L *is a* K*-quasiconformal curve, if* $K_L \le K$.

***Remark* 3.1.** It is well-known that if we are not interested with the coefficients of quasiconformality of the curve, then the definitions of "quasicircle" and "quasiconformal curve" are identical. But, if we are also interested with the coefficients of quasiconformality of the given curve, then we will consider, if the curve L is K-quasiconformal, then it is κ-quasicircle with $\kappa = \dfrac{K^2-1}{K^2+1}$.

By Remark 3.1, for simplicity, we will use both terms, depending on the situation.

Lemma 3.1 [1]. *Let* L *be a* K*-quasiconformal curve,* $z_1 \in L,\ z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq \preceq d(z_1, L_{r_0})\},\ w_j = \Phi(z_j),\ j = 1, 2, 3.$ *Then*

a) *The statements* $|z_1 - z_2| \preceq |z_1 - z_3|$ *and* $|w_1 - w_2| \preceq |w_1 - w_3|$ *are equivalent.*
So are $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$ *also equivalent.*

b) *If* $|z_1 - z_2| \preceq |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\varepsilon < 1,\ c > 1,\ 0 < r_0 < 1$ *are constants, depending on* G *and* $L_{r_0} := \{z = \psi(w) : |w| = r_0\}$.

Lemma 3.2. *Let* $G \in Q(\kappa)$ *for some* $0 \le \kappa < 1$. *Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+\kappa}$$

for all $w_1, w_2 \in \overline{\Delta}$.

This fact follows by [24] (Lemma 9.9) and the estimation for the Ψ is given as follows (see, for example, [12], Theorem 2.8):

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{3.1}$$

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h(z)$ is defined in (1.1).

Lemma 3.3 [8]. *Let* L *be a rectifiable Jordan curve;* $h(z)$ *is defined in* (1.1). *Then, for arbitrary* $P_n(z) \in \wp_n$, *any* $R > 1$ *and* $n \in \mathbb{N}$, *we have*

$$\|P_n\|_{\mathcal{L}_p(h, L_R)} \le R^{n + \frac{1+\gamma*}{p}} \|P_n\|_{\mathcal{L}_p(h, L)}, \quad p > 0, \tag{3.2}$$

where $\gamma^* = \max\left\{0; \gamma_k, k = \overline{1, m}\right\}$.

***Remark* 3.2.** In case $h(z) \equiv 1$, the estimation (3.2), has been proved in [14].

4. Proof of Theorem 2.1. Suppose that $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}$, for some $\frac{1}{2} \leq \beta_i \leq \alpha \leq 1$, $i = \overline{1,m}$, be given and $h(z)$ defined in (1.1). For given $R > 1$, lets $R_1 = 1 + \frac{R-1}{2}$, and let $\{\xi_j\}$, $1 \leq j \leq m \leq n$, be the zeros of $P_n(z)$ lying on Ω. Lets define the function Blashke with respect to the zeros of the polynomial $P_n(z)$:

$$B_m(z) := \prod_{j=1}^{m} B^j(z) := \prod_{j=1}^{m} \frac{\Phi(z) - \Phi(\xi_j)}{1 - \overline{\Phi(\xi_j)}\Phi(z)}\ ,\ z \in \Omega.$$

It is easy that the $B_m(\xi_j) = 0$ and $|B_m(z)| \equiv 1$ at $z \in L$. For any $p > 0$ and $z \in \Omega$ let us set

$$g_n(w) := \prod_{j=1}^{m} \left[\frac{\Psi(w) - \Psi(w_j)}{w}\right]^{p\mu_j/2} \left[\frac{P_n\left(\Psi(w)\right)}{w^{n+1}B_m(\Psi(w))}\right]^{p/2}, \quad w = \Phi(z). \tag{4.1}$$

The function $g_n(w)$ is analytic in Δ, continuous on $\overline{\Delta}$, $g_n(\infty) = 0$ and does not have zeros in Ω. We take an arbitrary continuous branch of the $g_n(w)$ and for this branch, we maintain the same designation. Then the Cauchy integral representation for the $g_n(w)$ given as

$$g_n(w) = -\frac{1}{2\pi i} \int\limits_{|\tau|=R_1} g_n(\tau) \frac{d\tau}{\tau - w},\ |w| = R.$$

Therefore,

$$\left| \prod_{j=1}^{m} \left[\frac{\Psi(w) - \Psi(w_j)}{w}\right]^{p\mu_j/2} \left[\frac{P_n\left(\Psi(w)\right)}{w^{n+1}B^j(\Psi(w))}\right]^{p/2} \right| \leq$$

$$\leq \frac{1}{2\pi} \int\limits_{|\tau|=R_1} \prod_{j=1}^{m} \left|\frac{\Psi(\tau) - \Psi(w_j)}{\tau}\right|^{p\mu_j/2} \left|\frac{P_n\left(\Psi(\tau)\right)}{\tau^{n+1}B^j(\Psi(\tau))}\right|^{p/2} \frac{|d\tau|}{|\tau - w|},$$

or

$$J_n := \prod_{j=1}^{m} \left[|\Psi(w) - \Psi(w_j)|\right]^{p\mu_j/2} \left|P_n\left(\Psi(w)\right)\right|^{p/2} \leq$$

$$\leq \frac{1}{2\pi} \prod_{j=1}^{m} \frac{\max_{|w|=R} |w|^{p\mu_j/2} \left|w^{n+1}B^j(\Psi(w))\right|^{p/2}}{\min_{|\tau|=R_1} |\tau|^{p\mu_j/2} \left|\tau^{n+1}B^j(\Psi(\tau))\right|^{p/2}} \times$$

$$\times \int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{p\mu_j^*/2} \left|P_n\left(\Psi(\tau)\right)\right|^{p/2} \frac{|d\tau|}{|\tau - w|}. \tag{4.2}$$

Since $\left|B^j(\zeta)\right| = 1$, for $\zeta \in L$, then for arbitrary ε, $0 < \varepsilon < \varepsilon_1$, there exists a circle $|w| = 1 + \frac{\varepsilon}{n}$, such that for any $j = 1, 2, \ldots, m$, the following are satisfied:

$$\left|B^j(\Psi(w)\right| > 1 - \varepsilon.$$

Then

$$|B_m(\zeta)| > (1-\varepsilon)^m \succeq 1$$

for $\varepsilon \le n^{-1}$ and $\zeta \in L_{R_1}$. Later

$$|\Phi(\zeta)| = R_1 > 1,\ |\Phi(\zeta)|^{n+1} = R_1^{n+1} \succeq 1$$

for $\zeta \in L_{R_1}$. On the other hand, we obtain

$$|w|^{p\mu_j/2} \preceq 1, \quad \left|w^{n+1} B_m(\Psi(w))\right|^{p/2} \preceq 1,\ z \in L_R.$$

According to this estimations, from (4.2), we have

$$J_n \preceq \int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j/2} \left|P_n\left(\Psi(\tau)\right)\right|^{p/2} \frac{|d\tau|}{|\tau-w|}.$$

Multiplying the numerator and determinator of the integrand by

$$h^{1/2}(\Psi(\tau)) \left|\Psi'(\tau)\right| = \left(h_0(\Psi(\tau)) \left|\Psi'(\tau)\right|\right)^{1/2} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{\gamma_j/2},$$

and applying the Hölder inequality, we obtain

$$J_n \preceq \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{\gamma_j} \left|P_n\left(\Psi(\tau)\right)\right|^{p} \left|\Psi'(\tau)\right| |d\tau| \right)^{1/2} \times$$

$$\times \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \frac{|d\tau|}{|\Psi'(\tau)|\,|\tau-w|^2} \right)^{1/2} =: J_{n,1} \times J_{n,2}, \qquad (4.3)$$

where

$$J_{n,1} := \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{\gamma_j} \left|P_n\left(\Psi(\tau)\right)\right|^{p} \left|\Psi'(\tau)\right| |d\tau| \right)^{1/2},$$

$$J_{n,2} := \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \frac{|d\tau|}{|\Psi'(\tau)|\,|\tau-w|^2} \right)^{1/2}.$$

By replacing the variable $\tau = \Phi(\zeta)$ and according to Lemma 3.3, we get

$$J_{n,1} \preceq \|P_n\|_{\mathcal{L}_p}^{p/2}. \qquad (4.4)$$

By applying (3.1), for all $z \in L_R$, we have

$$(J_{n,2})^2 = \int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \, \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2}. \tag{4.5}$$

Then, from (4.2) – (4.4), we get

$$\prod_{j=1}^{m} [|\Psi(w)-\Psi(w_j)|]^{\mu_j} \, |P_n(\Psi(w))| \preceq$$

$$\preceq \|P_n\|_{\mathcal{L}_p} \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \, \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \right)^{1/p}.$$

By denoting last integral as

$$\widetilde{J}_{n,m} := \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \, \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \right)^{1/p}, \tag{4.6}$$

we see that, to prove the theorem sufficiently estimation of the integral $\widetilde{J}_{n,m}$. To estimate the integral $\widetilde{J}_{n,m}$, we introduce

$$w_j := \Phi(z_j), \quad \varphi_j := \arg w_j, \quad L^j := L \cap \overline{\Omega}^j, \quad L_t^j := L_t \cap \overline{\Omega}^j,$$
$$F_t^j := \Phi(L_t^j), \quad t > 1, \quad j = \overline{1,m}, \tag{4.7}$$

where $\Omega_t^j := \Psi(\Delta'_{t,j})$;

$$\Delta'_{t,1} := \left\{ w = te^{i\theta} : t > 1, \ \frac{\varphi_m + \varphi_1}{2} \le \theta < \frac{\varphi_1 + \varphi_2}{2} \right\},$$

$$\Delta'_{t,m} := \left\{ w = te^{i\theta} : t > 1, \ \frac{\varphi_{m-1} + \varphi_m}{2} \le \theta < \frac{\varphi_m + \varphi_1}{2} \right\},$$

and, for $j = \overline{2, m-1}$,

$$\Delta'_{t,j} := \left\{ w = te^{i\theta} : t > 1, \ \frac{\varphi_{j-1} + \varphi_j}{2} \le \theta < \frac{\varphi_j + \varphi_{j+1}}{2} \right\}.$$

Then, since the points $\{z_j\}_{j=1}^m \in L$ are distinct, according to notations, given in (4.7), for arbitrary fixed j_0, $1 \le j_0 \le m$, we get

$$\left(\widetilde{J}_{n,m}\right)^p = \sum_{i=1}^{m} \int\limits_{F_{R_1}^i} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \, \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \asymp$$

$$\asymp \sum_{i=1}^{m} \int\limits_{F_{R_1}^i} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \, \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} =: \sum_{i=1}^{m} \widetilde{J}_{n,j_0}^i(F_{R_1}^i), \tag{4.8}$$

where, for each subarc $l \subset F_R^i$, $\widetilde{J}_{n,j_0}^i(l)$ is denoted by

$$\widetilde{J}_{n,j_0}^i(l) := \int\limits_l |\Psi(\tau) - \Psi(w_{j_0})|^{p\mu_{j_0} - \gamma_{j_0}} \frac{(|\tau| - 1)\,|d\tau|}{d(\Psi(\tau), L)\,|\tau - w|^2}. \tag{4.9}$$

It remains to estimate the integrals $\widetilde{J}_{n,j_0}^i(L_{R_1})$ for each $i = \overline{1,m}$. For simplicity of our next calculations, we assume that

$$m = 1,\ j_0 = 1, \quad \mu := \mu_1; \quad s^* := s_1^*, \quad \gamma := \gamma_1, \quad \beta := \beta_1, \quad R = 1 + \frac{1}{n}. \tag{4.10}$$

In this situation, the integral $\widetilde{J}_{n,j_0}^i(F_{R_1}^1)$ can be written as:

$$\widetilde{J}_{n,1}^1(F_{R_1}^1) := \int\limits_{F_{R_1}^1} |\Psi(\tau) - \Psi(w_1)|^{p\mu - \gamma} \frac{(|\tau| - 1)\,|d\tau|}{d(\Psi(\tau), L)\,|\tau - w|^2}. \tag{4.11}$$

By setting

$$\begin{gathered} L_{R_1,1}^1 := L_{R_1}^1 \cap \Omega(z_1, \delta_1), \quad L_{R_1,2}^1 := L_{R_1}^1 \cap (\Omega(z_1, \delta_1^*) \backslash \Omega(z_1, \delta_1)), \\ L_{R_1,3}^1 := L_{R_1} \backslash (L_{R_1,1}^1 \cup L_{R_1,2}^1), \quad F_{R_1,i}^1 := \Phi(L_{R_1,i}^1), \quad i = 1,2,3, \quad F_{R_1}^1 = \bigcup_{i=1}^3 F_{R_1,i}^1, \\ L_{R,1}^1 := L_R^1 \cap D(z_1, \delta_1), \quad L_{R,2}^1 := L_R^1 \cap (D(z_1, \delta_1^*) \backslash D(z_1, \delta_1)), \\ L_{R,3}^1 := L_R \backslash (L_1^1 \cup L_2^1), \quad F_{R,i}^1 := \Phi(L_{R,i}^1), \quad i = 1,2,3, \quad F_R^1 = \bigcup_{i=1}^3 F_{R,i}^1. \end{gathered} \tag{4.12}$$

By taking into consideration these designations, from (4.11), we have

$$\begin{gathered} \widetilde{J}_{n,1}^1(F_{R_1}^1) = \int\limits_{F_{R_1}^1} |\Psi(\tau) - \Psi(w_1)|^{p\mu - \gamma} \frac{(|\tau| - 1)\,|d\tau|}{d(\Psi(\tau), L)\,|\tau - w|^2} = \\ = \sum_{i=1}^3 \int\limits_{F_{R_1,i}^1} |\Psi(\tau) - \Psi(w_1)|^{p\mu - \gamma} \frac{(|\tau| - 1)\,|d\tau|}{d(\Psi(\tau), L)\,|\tau - w|^2} =: \\ =: \sum_{i=1}^3 \widetilde{J}_{n,1}^1(L_{R_1,i}^1), \end{gathered} \tag{4.13}$$

where

$$\widetilde{J}_{n,1}^1(F_{R_1,i}^1) := \int\limits_{F_{R,i}^1} |\Psi(\tau) - \Psi(w_1)|^{p\mu - \gamma} \frac{(|\tau| - 1)\,|d\tau|}{d(\Psi(\tau), L)\,|\tau - w|^2}, \quad i = 1,2,3. \tag{4.14}$$

We consider the individual cases.

1. Let $z \in L^1_{R,1}$. Lets denote $\widetilde{\zeta} \in F^1_{R_1,1}$ such that $d(\Psi(\tau), L) = \left|\zeta - \widetilde{\zeta}\right|$ and $\widetilde{w} := \Phi(\widetilde{\zeta})$;

$$F^{1,1}_{R_1,j} := \left\{\tau \in F^1_{R_1,j} \colon\ |\Psi(\tau) - \Psi(w_1)| \le c_1 d(\Psi(\tau), L)\right\},$$
$$F^{1,2}_{R_1,j} := \left\{\tau \in F^1_{R_1,j} \colon c_1 d(\Psi(\tau), L) < |\Psi(\tau) - \Psi(w_1)| < \delta^*_1\right\}, \quad j = 1, 2.$$

1.1. Then

$$\widetilde{J}^1_{n,1}(F^1_{R_1,1}) = \widetilde{J}^1_{n,1}(F^{1,1}_{R_1,1}) + \widetilde{J}^1_{n,1}(F^{1,2}_{R_1,1}),$$

and for $\widetilde{J}^1_{n,1}(F^{1,1}_{R_1,1})$ we have

$$\widetilde{J}^1_{n,1}(F^{1,1}_{R,1}) = \int\limits_{F^{1,1}_{R_1,1}} |\Psi(\tau) - \Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau| - 1)\,|d\tau|}{d(\Psi(\tau), L)\,|\tau - w|^2} \preceq$$

$$\preceq (R_1 - 1) \int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(\widetilde{w})|^{1-p\mu+\gamma}\,|\tau - w|^2} \preceq$$

$$\preceq (R_1 - 1) \int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\tau - \widetilde{w}|^{\frac{1-p\mu+\gamma}{\beta}}\,|\tau - w|^2} \preceq$$

$$\preceq (R_1 - 1)^{1-\frac{1-p\mu+\gamma}{\beta}} \int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\tau - w|^2} \preceq (R_1 - 1)^{1-\frac{1-p\mu+\gamma}{\beta}} \frac{1}{R - R_1} \preceq n^{\frac{1}{\alpha}}. \tag{4.15}$$

Now, lets estimate the integral $\widetilde{J}^1_{n,1}(F^{1,2}_{R,1})$. According to Lemma 3.1, for $\zeta \in \widetilde{J}^1_{n,1}(F^{1,2}_{R,1})$ we have $|\tau| - 1 < |\tau - w_1| \preceq 1$. We set $\varepsilon_0 := |\tau| - 1$. In this case, we take the discs centered at the point w_1, and radius $2^s\varepsilon_0$, $s = 1, 2, \ldots, N$, where we choose a number N such that the circle is $Q_N = \left\{\tau \colon\ |\tau - w_1| = 2^N\varepsilon_0\right\}$, that satisfies the conditions $Q_N \cap \{t \colon\ |t| = R\} \ne \varnothing$, $Q_{N+1} \cap \{t \colon\ |t| = R_1\} = \varnothing$. Then, setting $F^s_{R_1,1} := F^{1,1}_{R_1,1} \cap \left\{t \colon 2^{s-1}\varepsilon_0 \le |t - w_1| \le 2^s\varepsilon_0\right\}$, we have

$$\widetilde{J}^1_{n,1}(F^{1,1}_{R_1,1}) = \int\limits_{F^{1,1}_{R_1,1}} |\Psi(\tau) - \Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau| - 1)\,|d\tau|}{d(\Psi(\tau), L)\,|\tau - w|^2} =$$

$$= \int\limits_{F^{1,1}_{R_1,1}} \left|\frac{\Psi(\tau) - \Psi(w_1)}{\Psi(\tau) - \Psi(\widetilde{w})}\right| \frac{1}{|\Psi(\tau) - \Psi(w_1)|^{1-p\mu+\gamma}} \frac{(|\tau| - 1)\,|d\tau|}{|\tau - w|^2} \preceq$$

$$\preceq \sum_{s=1}^{\infty} \int\limits_{F^s_{R_1,1}} \left[\frac{|\tau - w_1|}{|\tau| - 1}\right]^{\varepsilon} \frac{|d\tau|}{|\tau - w_1|^{\frac{1-p\mu+\gamma}{\beta}}} \frac{(|\tau| - 1)\,|d\tau|}{|\tau - w|^2} \preceq$$

$$\preceq \sum_{s=1}^{\infty} \left(\frac{2^s\varepsilon_0}{\varepsilon_0}\right)^{\varepsilon} \frac{\varepsilon_0}{(2^{s-1}\varepsilon_0)^{\frac{1}{\alpha}}} \int\limits_{F^s_{R_1,1}} \frac{|d\tau|}{|\tau - w|^2} \preceq$$

$$\preceq 2^{\varepsilon}\varepsilon_0^{1-\frac{1}{\alpha}}\sum_{s=1}^{\infty}\left(\frac{2^{\varepsilon}}{2^{\frac{1}{\alpha}}}\right)^{s-1}\int\limits_{F^s_{R_1,1}}\frac{|d\tau|}{|\tau-w|^2}\preceq n\varepsilon_0^{1-\frac{1}{\alpha}}\sum_{s=1}^{\infty}\left(\frac{2^{\varepsilon}}{2^{\frac{1}{\alpha}}}\right)^{s-1}=$$

$$=nn^{\frac{1}{\alpha}-1}\sum_{s=1}^{\infty}\left(\frac{1}{2^{\frac{1}{\alpha}}}\right)^{s-1}\preceq n^{\frac{1}{\alpha}}. \tag{4.16}$$

1.2. For any $\zeta\in L^1_{R_1,2}$ and $z\in L^1_{R,1}$, $|\zeta-z_1|\geq\delta_1$ and, from Definition 2.4, we obtain

$$\left|\zeta-\widetilde{\zeta}\right|^{\alpha}\succeq\max\left\{|\zeta-z_1|^{\alpha-\beta_1};\ \left|\widetilde{\zeta}-z_1\right|^{\alpha-\beta_1}\right\}|\widetilde{w}-\tau|\geq$$

$$\geq\delta_1^{\alpha-\beta}|\widetilde{w}-\tau|\succeq|\widetilde{w}-\tau|.$$

Then, for this case, we get

$$\widetilde{J}^1_{n,1}(F^1_{R_1,2})=\int\limits_{F^1_{R_1,2}}|\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma}\frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2}\preceq$$

$$\preceq(\delta_1^*)^{p\mu-\gamma}\int\limits_{F^1_{R_1,2}}\frac{(|\tau|-1)\,|d\tau|}{|\widetilde{w}-\tau|^{\frac{1}{\alpha}}\,|\tau-w|^2}\preceq$$

$$\preceq\frac{(\delta_1^*)^{p\mu-\gamma}}{n}\frac{1}{(R_1-1)^{\frac{1}{\alpha}}}\int\limits_{F^1_{R,2}}\frac{|d\tau|}{|\tau-w|^2}\preceq$$

$$\preceq n^{\frac{1}{\alpha}-1}\int\limits_{F^1_{R,2}}\frac{|d\tau|}{|\tau-w|^2}\preceq nn^{\frac{1}{\alpha}-1}=n^{\frac{1}{\alpha}}. \tag{4.17}$$

1.3. For any $\zeta\in L^1_{R_1,3}$ and $z\in L^1_{R,1}$, $|\zeta-z_1|\geq\delta_1^*$ and $|\zeta-z|\geq\delta_1^*-\delta_1$. Then we obtain

$$\widetilde{J}^1_{n,1}(F^1_{R_1,3})=\int\limits_{F^1_{R_1,3}}|\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma}\frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2}\preceq$$

$$\preceq\frac{(2\operatorname{diam}\overline{G})^{p\mu-\gamma}}{(\delta_1^*-\delta_1)^2}(R_1-1)\int\limits_{F^1_{R_1,3}}\frac{|d\tau|}{\left|\zeta-\widetilde{\zeta}\right|}\preceq$$

$$\preceq(R_1-1)\int\limits_{F^1_{R_1,3}}\frac{|d\tau|}{|\tau-\widetilde{w}|^{\frac{1}{\alpha}}}\preceq(R_1-1)n^{\frac{1}{\alpha}-1}\preceq 1. \tag{4.18}$$

2. Let $z\in L^1_{R,2}$.

2.1. By setting

$$\widetilde{J}^1_{n,1}(F^1_{R_1,1})=\widetilde{J}^1_{n,1}(F^{1,1}_{R,1})+\widetilde{J}^1_{n,1}(F^{1,2}_{R_1,1}), \tag{4.19}$$

for $\widetilde{J}^1_{n,1}(F^{1,1}_{R,1})$, we have

$$\widetilde{J}^1_{n,1}(F^{1,1}_{R_1,1}) = \int\limits_{F^{1,1}_{R_1,1}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma}\,\frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \preceq$$

$$\preceq (R_1-1)\int\limits_{F^{1,1}_{R_1,1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(\widetilde{w})|^{1-p\mu+\gamma}\,|\tau-w|^2} \preceq$$

$$\preceq (R_1-1)\int\limits_{F^{1,1}_{R_1,1}} \frac{|d\tau|}{|\tau-\widetilde{w}|^{\frac{1-p\mu+\gamma}{\beta}}\,|\tau-w|^2} \preceq$$

$$\preceq (R_1-1)^{1-\frac{1-p\mu+\gamma}{\beta}}\int\limits_{F^{1,1}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^2} \preceq$$

$$\preceq (R_1-1)^{1-\frac{1-p\mu+\gamma}{\beta}}\,\frac{1}{R-R_1} \preceq n^{\frac{1}{\alpha}}. \tag{4.20}$$

Analogously, to prove the integral $\widetilde{J}^1_{n,1}(F^{1,2}_{R,1})$, we get

$$\widetilde{J}^1_{n,1}(F^{1,2}_{R,1}) \preceq n^{\frac{1}{\alpha}}. \tag{4.21}$$

2.2. Since, in this case

$$\left|\zeta-\widetilde{\zeta}\right|^{\alpha} \succeq \max\left\{|\zeta-z_1|^{\alpha-\beta_1};\ \left|\widetilde{\zeta}-z_1\right|^{\alpha-\beta_1}\right\}|\widetilde{w}-\tau| \geq$$

$$\geq \delta_1^{\alpha-\beta}\,|\widetilde{w}-\tau| \succeq |\widetilde{w}-\tau|,$$

then we get

$$\widetilde{J}^1_{n,1}(F^{1}_{R_1,2}) = \int\limits_{F^{1}_{R_1,2}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma}\,\frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \preceq$$

$$\preceq (\delta_1^*)^{p\mu-\gamma}\,(R_1-1)\int\limits_{F^{1}_{R_1,2}} \frac{|d\tau|}{|\tau-\widetilde{w}|^{\frac{1}{\alpha}}\,|\tau-w|^2} \preceq$$

$$\preceq (R_1-1)n^{\frac{1}{\alpha}}\int\limits_{F^{1}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^2} \preceq (R_1-1)n^{\frac{1}{\alpha}}n \preceq n^{\frac{1}{\alpha}}. \tag{4.22}$$

2.3. We have

$$\widetilde{J}^1_{n,1}(F^{1}_{R_1,3}) = \int\limits_{F^{1}_{R_1,3}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma}\,\frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \preceq$$

$$\preceq (2\mathrm{diam}\,\overline{G})^{p\mu-\gamma}(R_1-1)\int\limits_{F^1_{R,3}}\frac{|d\tau|}{|\tau-\widetilde{w}|^{\frac{1}{\alpha}}\,|\tau-w|^2}\preceq$$

$$\preceq (R_1-1)^{1-\frac{1}{\alpha}}\int\limits_{F^1_{R,3}}\frac{|d\tau|}{|\tau-w|^2}\preceq (R_1-1)^{1-\frac{1}{\alpha}}n\preceq n^{\frac{1}{\alpha}}. \tag{4.23}$$

3. Let $z\in L^1_3$.

3.1. Analogously to previously cases, we get

$$\widetilde{J}^1_{n,1}(F^1_{R_1,1})=\int\limits_{F^1_{R_1,1}}|\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma}\frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2}\preceq$$

$$\preceq\frac{(\delta_1)^{p\mu-\gamma}}{(\delta_1^*-\delta_1)^2}(R_1-1)\int\limits_{F^1_{R_1,1}}\frac{|d\tau|}{d(\Psi(\tau),L)}\preceq$$

$$\preceq\frac{1}{n}\int\limits_{F^1_{R_1,1}}\frac{|d\tau|}{|\tau-\widetilde{w}|^{\frac{1}{\alpha}}}\preceq n^{\frac{1}{\alpha}-2}. \tag{4.24}$$

3.2. We have

$$\widetilde{J}^1_{n,1}(F^1_{R_1,2})=\int\limits_{F^1_{R_1,2}}|\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma}\frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2}\preceq$$

$$\preceq(\delta_1^*)^{p\mu-\gamma}\frac{(R_1-1)}{(R-R_1)^2}\int\limits_{F^1_{R_1,,2}}\frac{|d\tau|}{|\tau-\widetilde{w}|^{\frac{1}{\alpha}}}\preceq$$

$$\preceq\frac{1}{(R-R_1)}n^{\frac{1}{\alpha}-1}\preceq n^{\frac{1}{\alpha}}. \tag{4.25}$$

3.3. We get

$$\widetilde{J}^1_{n,1}(F^1_{R_1,3})=$$

$$=\int\limits_{F^1_{R_1,3}}|\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma}\frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2}\leq$$

$$\leq\left(2\mathrm{diam}\,\overline{G}\right)^{p\mu-\gamma}(R_1-1)\frac{1}{(R-R_1)^2}\int\limits_{F^1_{R_1,3}}\frac{|d\tau|}{|\tau-\widetilde{w}|^{\frac{1}{\alpha}}}\preceq$$

$$\preceq\frac{1}{(R-R_1)}n^{\frac{1}{\alpha}-1}\preceq n^{\frac{1}{\alpha}}. \tag{4.26}$$

Combining the relations (4.6)–(4.26), for arbitrary $z\in L_R$, we obtain

$$|z-z_1|^{\mu}\,|P_n(z)|\ \preceq n^{\frac{1}{\alpha p}}\,\|P_n\|_{\mathcal{L}_p},\quad p>0. \tag{4.27}$$

The estimation (4.27) satisfied on L_R. We show that it is also carried out on L. For $R > 1$, let $w = \varphi_R(z)$ denotes the univalent conformal mapping of G_R onto B normalized by $\varphi_R(0) = 0$, $\varphi'_R(0) > 0$, and let $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, zeros of $P_n(z)$, lying on G_R. Let

$$B_{m,R}(z) := \prod_{j=1}^{m} B_R^j(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \overline{\varphi_R(\zeta_j)}\varphi_R(z)} \tag{4.28}$$

denotes a Blashke function with respect to zeros $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, of $P_n(z)$. Clearly,

$$|B_{m,R}(z)| \equiv 1, \ z \in L_R; \ |B_{m,R}(z)| < 1, \ z \in G_R.$$

For any $\mu > 0$ and $z \in G_R$, let us set

$$H_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{1/\mu}.$$

The function $H_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. Then, applying maximal modulus principle to $[H_n(z)]^{1/\mu}(z - z_1)$, we have

$$\left|\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{1/\mu}(z - z_1)\right| \leq \max_{\zeta \in \overline{G_R}} \left|\left[\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right]^{1/\mu}(\zeta - z_1)\right| \leq$$

$$\leq \max_{\zeta \in L_R} |P_n(\zeta)|^{1/\mu} |\zeta - z_1| \preceq \left(n^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p}\right)^{1/\mu}, \ z \in L,$$

and, therefore, we find

$$|(z - z_1)^{\mu} P_n(z)| \preceq n^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p}, \ z \in L.$$

Since the system of points $\{z_j\}_{j=1}^m$ are isolated and according to assumption (4.10), we get

$$\max_{z \in L} \left(\prod_{j=1}^{m} [|z - z_j|]^{\mu_j} |P_n(z)|\right) \preceq n^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p}, \quad p > 0,$$

and the proof (2.6) is completed.

Lets now we prove (2.7). For each $R > 1$, $p > 0$ and $z \in G_R$, let us set

$$T_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}$$

where $B_{m,R}(z)$ is a Blashke function defined in (4.28). The function $T_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_n(z)$ and for this branch we maintain the same designation. Then the Cauchy integral representation for the $T_n(z)$ in G_R gives

$$T_n(z) = \frac{1}{2\pi i} \int\limits_{L_R} T_n(\zeta) \frac{d\zeta}{\zeta - z}, \ z \in G_R,$$

or

$$\left|\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}\right| \leq \frac{1}{2\pi}\int\limits_{L_R}\left|\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta-z|} \leq \int\limits_{L_R}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta-z|},$$

since $|B_{m,R}(\zeta)|=1$, for $\zeta\in L_R$. Lets now $z\in L$. Multiplying the numerator and determinator of the integrand by $h^{1/2}(\zeta)$, by the Hölder inequality, we obtain

$$\left|\frac{P_n(z)}{B_{m,R}(z)}\right|^{p/2} \leq \frac{1}{2\pi}\left(\int\limits_{L_R} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{1/2}\times$$

$$\times\left(\int\limits_{L_R}\frac{|d\zeta|}{\prod_{j=1}^m|\zeta-z_j|^{\gamma_j}\,|\zeta-z|^2}\right)^{1/2} =: \frac{1}{2\pi}J_{n,1}(L_R)\times J_{n,2}(L_R).$$

Then, since $|B_{m,R}(z)|<1$, for $z\in L$, from Lemma 3.3, we have

$$|P_n(z)| \preceq (J_{n,1}(L_R)J_{n,2}(L_R))^{2/p} \preceq \|P_n\|_p\,(J_{n,2}(L_R))^{2/p}, \quad z\in L.$$

By using notations (4.7), for the integral $J_{n,2}$, we get

$$(J_{n,2}(L_R))^2 = \sum_{i=1}^m\int\limits_{L_R^i}\frac{|d\zeta|}{\prod_{j=1}^m|\zeta-z_j|^{\gamma_j}\,|\zeta-z|^2} \asymp \sum_{i=1}^m\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{\gamma_i}\,|\zeta-z|^2} =: \sum_{i=1}^m J_{n,2}^i(L_R^i),$$

where

$$J_{n,2}^i(L_R^i) := \int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{\gamma_i}\,|\zeta-z|^2}, \quad i=\overline{1,m},$$

since the points $\{z_j\}_{j=1}^m\in L$ are distinct. Therefore, it remains to estimate the integrals $J_{n,2}^i(L_R^i)$ for each $i=\overline{1,m}$. Setting $z=z_1$, and assume that $m=1$, under the notations (4.12), we have

$$|P_n(z_1)| \preceq \|P_n\|_{\mathcal{L}_p}\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{2+\gamma_1}} =$$

$$= \|P_n\|_{\mathcal{L}_p}\left[\int\limits_{L_{R,1}^1}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}} + \int\limits_{L_{R,2}^1}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}}\right]. \tag{4.29}$$

By applying (4.7), we obtain

$$\int\limits_{L_{R,1}^1}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}} = \int\limits_{F_{R,1}^1}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{2+\gamma_1}\,(|\tau|-1)} \preceq$$

$$\preceq \int\limits_{F^1_{R,1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{1+\gamma_1}(|\tau|-1)} \preceq n \int\limits_{F^1_{R,1}} \frac{|d\tau|}{|\tau-w_1|^{(\gamma_1+1)(2-\nu_1)}} \preceq n^{\frac{\gamma_1+1}{\beta_1}},$$

$$\int\limits_{L^1_{R,2}} \frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}} \preceq (\delta)^{-2-\gamma_1} \operatorname{mes} L^1_{R_1,1} \preceq 1.$$

Then, from (4.29), we get

$$|P_n(z_1)| \preceq n^{\frac{\gamma_1+1}{p\beta_1}} \|P_n\|_{\mathcal{L}_p},$$

and, according to our assumption $m=1$, we complete the proof of estimation (2.7).

Theorem 2.1 is proved.

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